MATerials MATemàtics
Volum 2009, treball no. 6, 26 pp. ISSN: 1887-1097
Publicació electrònica de divulgació del Departament de Matemàtiques
de la Universitat Autònoma de Barcelona
www.mat.uab.cat/matmat

Values of the Riemann zeta function at integers

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1 Introduction

The Riemann zeta function is one of the most important and fascinating functions in mathematics. It is very natural as it deals with the series of powers of natural numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}, \qquad \text{etc.} \tag{1}$$

Leonhard Euler

Originally the function was defined for real arguments as

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad \text{for} \quad x > 1. \tag{2}$$

It connects by a continuous parameter all series from (1). In 1734 Leonhard Euler (1707 - 1783) found something amazing; namely he determined all values $\zeta(2), \zeta(4), \zeta(6), \ldots$ – a truly remarkable discovery. He also found a beautiful relationship between prime numbers and $\zeta(x)$ whose significance for current mathematics cannot be overestimated. It was Bernhard Riemann (1826 - 1866), however, who recognized the importance of viewing $\zeta(s)$ as a function of a complex variable $s = x + iy$ rather than a real variable x. Moreover, in 1859 Riemann gave a formula for a unique (the so-called holomorphic) extension of the function onto the entire complex plane $\mathbb{C}$ except $s = 1$. However, the formula (2) cannot be applied anymore if the real part of s, $\operatorname{Re} s = x$ is ≤ 1. It will be discussed more precisely in §4.

Even after more than two hundred years of study, the Riemann zeta function is as mysterious as ever. For instance, except for the so-called trivial zeros at $-2, -4, -6, \dots$, the position of the other zeros is still an open conjecture. It is a subject of the *Riemann Hypothesis*. No unsolved conjecture is more celebrated nor more desirable than the Riemann Hypothesis.

Bernhard Riemann

Another problem which we consider and which is the main goal of this note is the structure of values $\zeta(n)$, where $n = 0, \pm 1, \pm 2, \dots$. Some values can be computed explicitly, but others, $\zeta(2k+1)$, where $k = 1, 2, \dots$, are still mysterious. Even the question whether $\zeta(2k+1)$ is a rational number, is solved only for the value $\zeta(3)$.

A brief discussion of references. There is an enormous amount of literature on the Riemann zeta function. Remarkably well written with only modest, necessary background are T.M. Apostol [3], K. Ireland and M. Rosen [14] and A. Weil [23]. J. Bruna [8], B.C. Berndt [6], G. Everest, C. Röttger and T. Ward [12], M.R. Murty and M. Reece [18] and A. Van der Poorten [22] contain excellent additional material related to our article approached in an elementary way. M. Abramowitz and I.A. Stegun [1] and H. Bateman (A. Erdelyi ed.) [5] contain a real treasure of information about the zeta function and related functions written concisely and informatively. A.B. Goncharov's [13] brilliant short article provides a glimpse into more advanced topics in current mathematics linked to the values of zeta functions in an extraordinary way. E. Landau [15] and A.I. Markushevich [16] are a good source of basic reference books in calculus and complex analysis for a deeper study of the analytic properties of the zeta function. E.C. Titchmarsh [21] is a true classic book on the Riemann zeta function with excellent end-of-chapter notes by D.R. Heath-Brown which update the second edition. This book, however, already requires a solid background in analysis. We hope that these suggestions about the bibliography will help the reader in his/her further exploration of the topics discussed in this paper.

2 Definition of the Riemann zeta function

In the formula (2) the variable x can be replaced by complex $s = x + iy$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \operatorname{Re} s = x > 1. \tag{3}$$

It is customary to denote complex arguments in the Riemann zeta function by the letter s, and in arbitrary functions by z or w. Real arguments are usually denoted by x and y, and the decomposition of a complex number into real and imaginary parts by $x + iy$. The set of complex numbers $z = x + iy$ will be denoted by $\mathbb{C}$ and geometrically it can be identified with a plane.

Some clarification is needed to explain what we mean by the complex power of a natural number. The expression n^s for $s \in \mathbb{C}$ is defined

$$n^s = e^{s \log n},$$

where log is the natural logarithm with base e and

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{for} \quad z \in \mathbb{C}.$$

Consequently we have

$$n^s = \sum_{k=0}^{\infty} \frac{s^k (\log n)^k}{k!}.$$

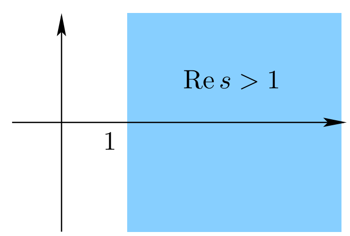


Figure 1: *The half-plane* Re $s > 1$

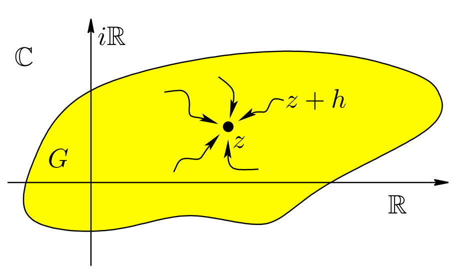


Figure 2: $z + h$ *approaches* z

Formally, the function $\zeta(s)$ from (3) is defined only for $\operatorname{Re} s > 1$ because otherwise the absolute value of n^s is too small and the series $\sum_{n=1}^{\infty} |1/n^s|$ diverges. However, this function is holomorphic in the half-plane $\operatorname{Re} s > 1$ as in Fig. 1.

By definition, a complex-valued function $f = f(z)$ defined in an open, connected set $G \subset \mathbb{C}$ (i.e., G is a domain) is *holomorphic* if it is differentiable in the complex sense in G, namely,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \quad z \in G,$$

exists.

Even that formally the definition of the derivative is the same as in the real case, the main difference is that the point $z+h$ can approach z from different directions as in Fig. 2. This means that this condition is extremely strong in comparison with the real variable case.

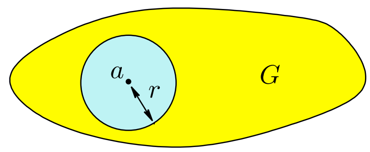


Figure 3: *Disc of convergence*

Another, equivalent definition of holomorphic functions is by expansion into power series. Namely a function $f = f(z)$, $z \in G$, is holomorphic in the domain G if for any $a \in G$ there exists a convergent power series such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{for} \quad |z-a| < r, \quad r > 0.$$

i.e., $f(z)$ is equal to the power series in the disc $\{|z-a| < r\}$ as in Fig. 3.

3 The Riemann zeta function in terms of prime numbers

As we hinted before, Euler found another formula for the zeta function, namely

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \text{for} \quad \operatorname{Re} s > 1, \tag{4}$$

where p runs through all primes $p = 2, 3, 5, 7, 11, \dots$. He thus proved the equivalence of both formulas (4) and (3). Here is the key idea of the proof. We start with the product

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \cdots$$

and write each factor as

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots .$$

In the next calculations we use the basic property that any natural number can be expressed uniquely (up to the order of factors) as a product of prime numbers. Taking the product over prime numbers $\leq N$ and denoting by P the greatest prime number that satisfies this inequality, we get

$$\begin{aligned}\prod_{p\leq N}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right) &= \\ &= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\dots\right)\dots\left(1+\frac{1}{P^s}+\frac{1}{P^{2s}}+\dots\right) \\ &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{2^{2s}}+\frac{1}{5^s}+\frac{1}{2^s3^s}+\frac{1}{7^s}+\dots \\ &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\dots+\frac{1}{N^s}+\text{remainder.}\end{aligned}$$

Since $\operatorname{Re} s > 1$, it is elementary calculus to show that the remainder can be made arbitrarily small if N is sufficiently large. When N will go to infinity, the formula as the product (4) becomes equal to the formula as the sum (3). Therefore, for $\zeta(s)$ we have two formulas: one in terms of series and another one in terms of product.

4 Extensions of holomorphic functions

One of the main properties of holomorphic functions is uniqueness in the sense that if two holomorphic functions f and g defined in a domain G are equal on a sequence $z_n \in G$, $\lim_{n\to\infty} z_n = z_0 \in G$, i.e., $f(z_n) = g(z_n)$ for $n = 1, 2, \dots$, then $f = g$ in G; see Fig. 4. Of course such a property is not true for functions in real calculus.

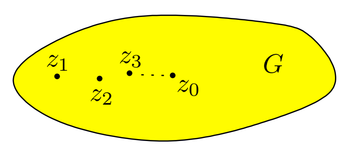


Figure 4: *Sequence of points*

In particular, if a holomorphic function f is defined in a domain $G_1 \subset \mathbb{C}$ and another holomorphic function g is defined in a domain $G_2 \subset \mathbb{C}$ with $G_1 \cap G_2 \neq \emptyset$ and $f = g$ on the intersection, then g is determined uniquely by f; see Fig. 5.

If a holomorphic function f is defined by a power series which converges in a disc and diverges outside that disc, it does not mean that the function

f cannot be holomorphically extended beyond this disc. A simple example:

$$f(z) = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

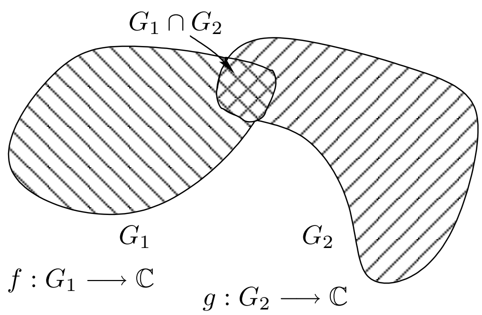


Figure 5: *Holomorphic extensions*

Obviously the series diverges for values $|z| \geq 1$. However, the function f can be holomorphically extended to the entire complex plane $\mathbb{C}$ except $z = 1$, by the formula

$$f(z) = \frac{1}{1-z}.$$

A natural question appears: whether the Riemann zeta function can be holomorphically extended beyond the half-plane $\operatorname{Re} s > 1$? The answer is yes, which we show in two steps. The first step is easy, the second more difficult.

4.1 Extension of $\zeta(s)$ from $\{\operatorname{Re} s > 1\}$ to $\{\operatorname{Re} s > 0\}$

Let us calculate

$$\begin{aligned}
(1 - 2^{1-s})\zeta(s) &= \left(1 - 2 \cdot \frac{1}{2^s}\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots\right) \\
&= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots\right) \\
&= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \dots \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}
\end{aligned}$$

We obtained another formula for $\zeta(s)$,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} \qquad \text{for} \quad \operatorname{Re} s > 0, \ s \neq 1,$$

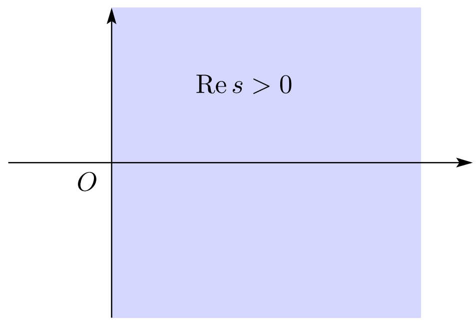


Figure 6: *The alternating series converges for* $\operatorname{Re} s > 0$

so the alternating series converges in a bigger half-plane (see Fig. 6) than the originally defined function $\zeta(s)$ in (3), but we have to remove $s = 1$ since the denominator $1 - 2^{1-s}$ vanishes there. This rather easy extension of $\zeta(s)$ from s with $\operatorname{Re} s > 1$ to s with $\operatorname{Re} s > 0$ is already significant as it allows us to formulate the Riemann Hypothesis about the zeros of $\zeta(s)$ in the critical strip (see Subsection 4.3).

4.2 Functional equation for the Riemann zeta function

The second step, which provides a holomorphic extension for $\zeta(s)$ from $\{\operatorname{Re} s > 0,\ s \neq 1\}$ to $\{\operatorname{Re} s < 0\}$, see Fig. 7, was proved by Riemann in 1859. We do not give a proof here of the so-called functional equation, but the proof can be found, e.g. in the book by Titchmarsh [21]. Alternatively one can first holomorphically extend $\zeta(s)$ step by step to half-planes $\{\operatorname{Re} s > k,\ s \neq 1\}$, where k is any negative integer. For details of this method, see for example the papers [11] and [12].

There are few versions of the functional equation; here we formulate two of them:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\pi s/2)\, \Gamma(s)\, \zeta(s)\ , \qquad \text{for} \quad \operatorname{Re} s > 0\ , \qquad (5)$$

$$\zeta(s) = 2(2\pi)^{s-1} \sin(\pi s/2)\, \Gamma(1-s)\, \zeta(1-s)\ , \qquad \text{for} \quad \operatorname{Re} s < 1\ , \qquad (6)$$

where

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \qquad \text{for} \qquad \operatorname{Re} s > 0. \qquad (7)$$

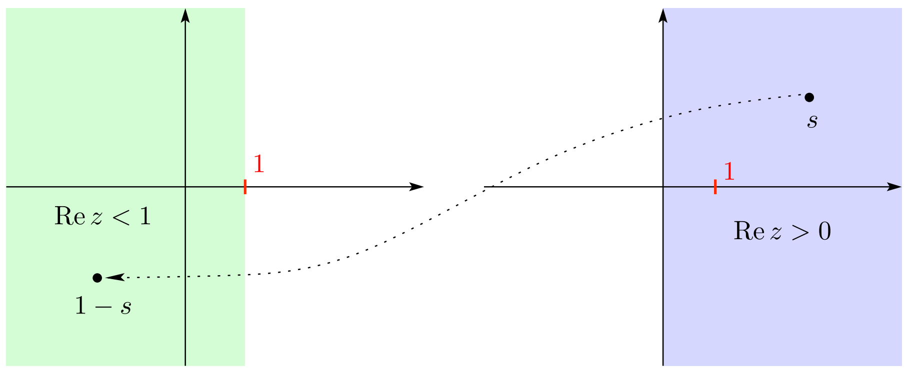


Figure 7: *Holomorphic extension domain from the right half-plane to the left half-plane*

Before we give more information about the function $\Gamma(s)$, we mention that each of the equations (5) and (6) give an extension of $\zeta(s)$ on the entire plane $\mathbb{C}$ except $s = 1$, as is illustrated in Fig. 7.

The gamma function was already known to Euler. It generalizes the factorial $n!$, namely

$$\Gamma(n) = (n-1)! \qquad \text{for} \quad n = 1, 2, \dots \ .$$

Its basic properties are that $\Gamma(z)$ is holomorphic on the entire plane $\mathbb{C}$ except for the points $z = 0, -1, -2, \dots$. At these points there are simple singularities, called poles, where we have the limits

$$\lim_{z \to -k} (z+k)\Gamma(z) = \frac{(-1)^k}{k!}, \quad k = 0, 1, 2, \dots \ .$$

From the definition of the gamma function (7) it is not clear that it can be extended onto the entire plane except for $z = 0, -1, -2, -3, \dots$, and that is non-vanishing. Fortunately there are other equivalent definitions of $\Gamma(z)$ from which these properties follow more easily; see [16]. Namely we have

$$\Gamma(z) = e^{-\gamma z} \left[z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \right]^{-1}, \quad \Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z+1) \cdot \ldots \cdot (z+n)}.$$

From the second formula for the gamma function we see that $\Gamma(z)$ is holomorphic and nonvanishing for $\operatorname{Re} z > 0$.

4.3 The Riemann Hypothesis

The Riemann Hypothesis is the most famous open problem in mathematics. Originally formulated by Riemann, David Hilbert then included the conjecture on his list of the most important problems during the Congress of Mathematicians in 1900, and recently the hypothesis found a place on the list of the Clay Institute's seven greatest unsolved problems in mathematics.

It follows from the formula of $\zeta(s)$ as the product (4) that the function does not vanish for $\operatorname{Re} s > 1$. Next, using the functional equation (6) and the fact that $\Gamma(z) \neq 0$ for $\operatorname{Re} z > 0$, we see that $\zeta(s)$ vanishes in the half-plane $\operatorname{Re} s < 0$ only at the points where the function sine is zero, namely from (6) we obtain

$$\zeta(-2k) = 2(2\pi)^{-2k-1} \underbrace{\sin(\pi(-2k)/2)}_{=0} \Gamma(1+2k)\,\zeta(1+2k) = 0, \; k = 1, 2, \ldots \;. \tag{8}$$

The above considerations do not tell us about the zeros of $\zeta(s)$ in the strip $0 < \operatorname{Re} s < 1$. Actually there are zeros in this strip and they are called nontrivial zeros. Calculation of some number of these nontrivial zeros shows that they are lying exactly on the line $\operatorname{Re} s = \frac{1}{2}$, called the *critical line*; see Fig. 8. Now with the help of computers it is possible to calculate an enormous number of zeros, currently at the level of 10^{13} (ten trillion). It is interesting to mention that before the computer era began roughly in the middle of the twentieth century, only about a thousand zeros were calculated. Of course all of these zeros are calculated with some (high) accuracy: they are lying on the critical line. However, there is no proof that really all nontrivial zeros lie on this line and this conjecture is called the *Riemann Hypothesis*.

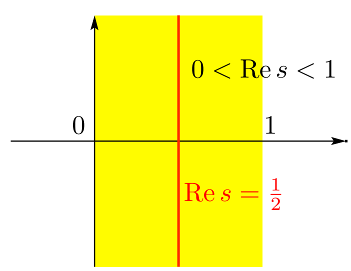


Figure 8: *Critical line*

Riemann Hypothesis: All nontrivial zeros are on the line $\operatorname{Re} s = \frac{1}{2}$.

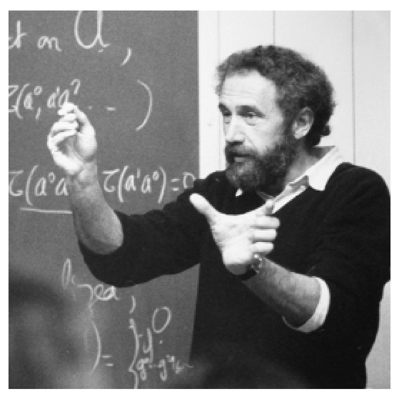
Many great mathematicians have contributed to a better understanding of the Riemann Hypothesis. There is no room to even partially list them here. Only we mention four of them: André Weil (1906 - 1998), Atle Selberg

(1917 - 2007), Enrico Bombieri (1940 -), and Alain Connes (1947 -). The last three received the Fields Medal (in 1950, 1974, and 1982, respectively), which is considered an equivalent to a Nobel Prize in mathematics. The Fields Medal is awarded only to scientists under the age of forty. If someone proves the Riemann Hypothesis and is relatively young, then they surely will receive this prize.

André Weil

Atle Selberg

Enrico Bombieri



Alain Connes

5 What is known about the values of $\zeta(s)$ at integers?

Here we sum up what is known about the values of the Riemann zeta function at integers. In Section 7 we discuss these statements more precisely.

- $\zeta(-2k) = 0$, $k = 1, 2, \ldots$, these are trivial zeros and are evaluated in (8).
- The values $\zeta(2k)$ for $k = 1, 2 \ldots$, have been found by L. Euler in 1734; we give a proof of these formulas in Section 7.
- $\zeta(-2k+1)$ for $k = 1, 2, \ldots$. When we know $\zeta(2k)$, then we can evaluate

$$\begin{aligned}\zeta(-2k+1) &= 2(2\pi)^{-2k}\sin(\pi(-2k+1)/2)\,\Gamma(2k)\,\zeta(2k)\\ &= 2(2\pi)^{-2k}(-1)^k(2k-1)!\,\zeta(2k).\end{aligned}$$

- $\zeta(2k+1)$??? for $k = 1, 2, \ldots$. There is a mystery about these values. Among the few known results is that $\zeta(3)$ is irrational (Apéry) and the results of Rivoal show that there are an infinite number of irrationals among them.
- $\zeta(0) = -\frac{1}{2}$ follows from (9) and (6).
- $\zeta(1)$ does not exist, but $\gamma =$ Euler's constant can be extracted:

$$\zeta(s) = \frac{1}{s-1} + \gamma + g(s-1), \quad \text{where} \quad g(s-1) \to 0 \text{ as } s \to 1. \tag{9}$$

  Amazingly, it is not known whether γ is rational or not.

6 Bernoulli numbers

6.1 An original way to evaluate Bernoulli numbers

Jacob Bernoulli

To explain some values of the Riemann zeta function at integers, we have to go back and learn about *Bernoulli numbers*, which were named after Jacob Bernoulli (1654 - 1705). He set a goal for himself to find a formula for the finite sum of powers of consecutive positive integers:

$$S_k(n) = 1^k + \ldots + (n-1)^k,$$

where $k = 1, 2, \ldots$, $n = 2, 3, \ldots$. It turned out that $S_k(n)$ is a polynomial in n of degree $k + 1$, as we will see in a moment. Bernoulli was successful in finding a general formula for these polynomials, and as he wrote in his book, *Ars Conjectandi*, that “in less than a half of a quarter of an hour he was able to sum the tenth powers of the first thousand integers” (see [14]).

There is a large amount of literature on Bernoulli numbers and their relations to various areas of mathematics (see the bibliography [9] on this subject between the years 1713 - 1990).

If k is small, then it is relatively easy to find formulas for $S_k(n)$, namely

$$
\begin{array}{llllllllllll}
S_1(n) & = & \frac{1}{2}n^2 & -\frac{1}{2}n & & & & & & & \\
S_2(n) & = & \frac{1}{3}n^3 & -\frac{1}{2}n^2 & +\frac{1}{6}n & & & & & & \\
S_3(n) & = & \frac{1}{4}n^4 & -\frac{1}{2}n^3 & +\frac{1}{4}n^2 & & & & & & \\
S_4(n) & = & \frac{1}{5}n^5 & -\frac{1}{2}n^4 & +\frac{1}{3}n^3 & & -\frac{1}{30}n & & & & \\
S_5(n) & = & \frac{1}{6}n^6 & -\frac{1}{2}n^5 & +\frac{5}{12}n^4 & & -\frac{1}{12}n^2 & & & & \\
S_6(n) & = & \frac{1}{7}n^7 & -\frac{1}{2}n^6 & +\frac{1}{2}n^5 & & -\frac{1}{6}n^3 & & +\frac{1}{42}n & & \\
S_7(n) & = & \frac{1}{8}n^8 & -\frac{1}{2}n^7 & +\frac{7}{12}n^6 & & -\frac{7}{24}n^4 & & +\frac{1}{12}n^2 & & \\
S_8(n) & = & \frac{1}{9}n^9 & -\frac{1}{2}n^8 & +\frac{2}{3}n^7 & & -\frac{7}{15}n^5 & & +\frac{2}{9}n^3 & -\frac{1}{30}n & \\
S_9(n) & = & \frac{1}{10}n^{10} & -\frac{1}{2}n^9 & +\frac{3}{4}n^8 & & -\frac{7}{10}n^6 & & +\frac{1}{2}n^4 & -\frac{3}{20}n^2 & \\
S_{10}(n) & = & \frac{1}{11}n^{11} & -\frac{1}{2}n^{10} & +\frac{5}{6}n^9 & & -n^7 & & +n^5 & -\frac{1}{2}n^3 & +\frac{5}{66}n \\
\dots & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \dots & & \\
 & & \frac{1}{k+1} & -\frac{1}{2} & \frac{k}{12} & 0 & -\frac{k(k-1)(k-2)}{720} & 0 & \dots & &
\end{array}
$$

So we have

$$S_k(n) = \frac{1}{k+1}n^{k+1} - \frac{1}{2}n^k + \frac{k}{12}n^{k-1} + 0\,n^{k-2} - \frac{k(k-1)(k-2)}{720}n^{k-3} + \cdots .$$

For technical reasons, it is better to adjust the coefficients of the polynomial $S_k(n)$ by the factor $\dfrac{1}{k+1}\dbinom{k+1}{j}$. Then we get

$$\begin{aligned} S_k(n) &= \frac{1}{k+1}\left[B_0 n^{k+1} + \binom{k+1}{1} B_1 n^k + \ldots + \binom{k+1}{k} B_k n\right] \\ &= \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j} B_j\, n^{k+1-j}\,. \end{aligned} \tag{10}$$

There are the recurrence relations for B_k:

$$\begin{aligned} &B_0 = 1, \\ &B_0 + 2B_1 = 0, \\ &B_0 + 3B_1 + 3B_2 = 0, \\ &B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ &\sum_{j=0}^{k}\binom{k+1}{j} B_j = 0,\ k \geq 1\,. \end{aligned}$$

These beautiful rational numbers are *Bernoulli numbers*.

6.2 An analytic way to evaluate Bernoulli numbers

Amazingly, we can evaluate Bernoulli numbers in a completely different way. Consider the holomorphic function

$$\frac{z}{e^z - 1} \quad \text{for} \quad |z| < 2\pi$$

and expand it into the power series

$$\frac{z}{e^z-1} = 1 - \frac{1}{2}z + \frac{1}{2!}\frac{1}{6}z^2 + \frac{1}{4!}(-\frac{1}{30})z^4 + \ldots = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \tag{11}$$

It appears that the coefficients B_n of the series above are exactly the same numbers which were defined in a completely different way.

The first Bernoulli numbers give a misleading impression that they converge to zero:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

Actually the opposite is true

$$(-1)^{k+1}B_{2k} > 0, \quad (-1)^{k+1}B_{2k} \sim \frac{2(2k)!}{(2\pi)^{2k}} \quad \text{as} \quad k \longrightarrow \infty\,.$$

The latter statement means that

$$\lim_{k\to\infty} \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!} = 1.$$

6.3 Bernoulli polynomials

We can define Bernoulli polynomials in terms of Bernoulli numbers:

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j\, x^{k-j},$$

or by expansion of the complex analytic function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k \qquad \text{for} \quad |z| < 2\pi\,. \tag{12}$$

It is not difficult to prove that the polynomials have nice recurrence relations

$$\begin{aligned} B_k(x+1) - B_k(x) &= kx^{k-1} && \text{if} \quad k \geq 1, \\ B_k(0) &= B_k(1) && \text{if} \quad k \geq 2. \end{aligned}$$

From the definition of the Bernoulli numbers (11) and polynomials (12), we immediately obtain

$$B_k = B_k(0) \quad \text{for all} \quad k \geq 1.$$

Using these properties we see that the sums $S_k(n)$ in (10) can be expressed in terms of Bernoulli polynomials:

$$S_k(n) = \frac{B_{k+1}(n) - B_{k+1}}{k+1} = \frac{B_{k+1}(n) - B_{k+1}(0)}{k+1} = \frac{B_{k+1}(n) - B_{k+1}(1)}{k+1}\,.$$

7 What is known about some values at integers?

7.1 Euler's calculations of $\zeta(2k)$

After several years of struggle, Leonhard Euler proved in 1734 a stunning formula for $\zeta(2k)$, where n is a natural number. For $|z| < \pi$ we have

$$\begin{aligned} z \cot z = z\frac{\cos z}{\sin z} &= 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2 - z^2} \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2} \frac{1}{1-\left(\frac{z}{n\pi}\right)^2} \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2} \sum_{k=0}^{\infty} \left(\frac{z}{n\pi}\right)^{2k} \\ &= 1 - 2\sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k+2}}\right) \frac{z^{2k+2}}{\pi^{2k+2}} \\ &= 1 - 2\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right) \frac{z^{2k}}{\pi^{2k}} \,. \end{aligned}$$

Another formula for $z \cot z$:

$$z \cot z = z\frac{\cos z}{\sin z} = iz\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = iz + \frac{2iz}{e^{2iz}-1} = 1 + \sum_{k=2}^{\infty} B_k \frac{(2iz)^k}{k!},$$

where B_k are Bernoulli numbers. Comparing the coefficients of powers of z^k from the above two calculations, we get the famous formula

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}.$$

Just the first few values:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}\,.$$

7.2 Are the relations between $\zeta(k)$ and B_k accidental?

What are the following sums? If we just formally substitute in formula (3) for $\zeta(s)$ arguments $s = 0, -1, -2, \dots$, ignoring the fact that our series diverges, we obtain:

$$
\begin{array}{llll}
1+1+1+\dots & = & =\zeta(0) & = -\frac{1}{2} \\
1+2+3+\dots & = & =\zeta(-1) & = -\frac{1}{12} \\
1^2+2^2+3^2+\dots & = & =\zeta(-2) & = 0 \\
\dots \quad \dots \quad \dots & \dots & \dots & \dots \\
1^k+2^k+3^k+\dots & = & =\zeta(-k) & = -\frac{B_{k+1}}{k+1} \,.
\end{array}
$$

The relations between $\zeta(k)$ and B_k are not accidental! In paper [17] we have the following calculations, where in $S_k(n)$ the argument n is replaced by a variable x and then the corresponding function is integrated over the interval [0, 1]:

$$1+\dots+1 = n-1 \;\rightsquigarrow\; x-1 \;\rightsquigarrow\; \int_0^1 (x-1)\,dx = -\frac{1}{2} = \zeta(0),$$

$$1+\dots+(n-1) = \frac{n(n-1)}{2} \;\rightsquigarrow\; \frac{x(x-1)}{2} \;\rightsquigarrow\; \int_0^1 \frac{x(x-1)}{2}\,dx = -\frac{1}{12} = \zeta(-1),$$

$$1^2+\dots+(n-1)^2 = \frac{n(n-1)(2n-1)}{6} \;\rightsquigarrow\; \frac{x(x-1)(2x-1)}{6} \;\rightsquigarrow$$
$$\rightsquigarrow\; \int_0^1 \frac{x(x-1)(2x-1)}{6}\,dx = 0 = \zeta(-2),$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$S_k(n) = 1^k+\dots+(n-1)^k \;\rightsquigarrow\; S_k(x) \;\rightsquigarrow\; \int_0^1 S_k(x)\,dx = -\frac{B_{k+1}}{k+1} = \zeta(-k)\,.$$

See also [18] and [12] for another nice variant of obtaining this relation in a very simple way.

7.3 What is known about $\zeta(2k+1)$?

One of the most spectacular discoveries about $\zeta(2k+1)$ was the proof by R. Apéry in 1978 that $\zeta(3)$ is irrational (see Apéry [2]). Apéry gave another formula for this value in terms of a series that converges quickly

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}}. \tag{13}$$

There is a theorem in number theory which says that if a series of rational numbers converges "*fast enough*", then the sum of the series is an irrational number. The series on the right of (13) satisfies the assumption of this theorem, therefore the conclusion is that $\zeta(3)$ is irrational.

There are also similar formulas for $\zeta(2)$ and $\zeta(4)$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}},$$

however explicit formulas for these values were already found by Euler, namely $\pi^2/6$ and $\pi^4/90$, respectively. Unfortunately there are no related formulas for other values $\zeta(n)$, $n = 5, 6, 7, \dots$, which give at least the irrationality of them. For a simple version of Apéry's proof, the reader is referred to a paper by Van der Poorten [22].

Relatively recently, other substantial, remarkable results were obtained by Tanguy Rivoal and others. We have

Theorem 1 (Rivoal, 2000, [19]) *There are infinitely many irrational values of the Riemann zeta function at odd positive integers. Moreover, if*

$$N(n) = \#\, \textit{irrational numbers among}\, \zeta(3), \zeta(5), \dots, \zeta(2n+1),$$

then

$$N(n) \geq C \log n \quad \textit{for } n \textit{ large, where } C \textit{ can be taken } \frac{1}{2(1+\log 2)}.$$

Theorem 2 (Rivoal, 2002, [20]) *At least one of the nine numbers* $\zeta(5)$, $\zeta(7), \dots, \zeta(21)$ *is irrational.*

A reviewer for Math. Reviews, Vadim Zudilin, of Rivoal's 2002 paper, improved the result:

Theorem 3 (V. Zudilin, 2001, [24]) *At least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.*

8 Relations between the values of the zeta function

The best result about the values $\zeta(2k+1)$, $k = 1, 2, \ldots$, would be if there are explicit formulas as in the case of $\zeta(2k)$. However, currently it seems a very difficult problem. Another significant result would be to give finite relations between the values $\zeta(2k+1)$, involving only a finite number of such values in a single relation, or show that there are no such relations. Again, it looks like a very difficult problem.

In the remaining part of this note, we describe a result which was obtained by the authors [10] giving infinite relations between $\zeta(2k+1)$. In the literature, one can find many infinite relations among these values. The significance of the result in [10] is in its universality: plugging a holomorphic function from a relatively large class of functions, we get an infinite relation between the values $\zeta(2k+1)$, $k = 1, 2, \ldots$.

8.1 The values of $\zeta(n)$ in terms of integrals

The following formulas are well-known in the literature:

$$\frac{1}{j^{2k}} = \frac{(-1)^{k-1}(2\pi)^{2k}}{(2k)!}\int_0^1 B_{2k}(t)\cos(2\pi jt)\,dt\ ,$$
$$\frac{1}{j^{2k+1}} = \frac{(-1)^{k-1}(2\pi)^{2k+1}}{(2k+1)!}\int_0^1 B_{2k+1}(t)\sin(2\pi jt)\,dt\ ,$$

where $j, k = 1, 2, \ldots$, and $k = 0$ in the second formula is allowed.

Then, using the standard summation formulas for sine and cosine

$$\frac{1}{2} + \cos\theta + \cos(2\theta) + \ldots + \cos(n\theta) = \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}$$

and

$$\sin\theta + \sin(2\theta) + \ldots + \sin(n\theta) = \frac{\cos(\theta/2) - \cos((n+1/2)\theta)}{2\sin(\theta/2)} ,$$

we obtain:

In the even case

$$\zeta(2k) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^{2k}} = \frac{(-1)^{k-1}(2\pi)^{2k}}{2\,(2k)!} \lim_{n\to\infty} \int_0^1 B_{2k}(t) \frac{\sin((2n+1)\pi t)}{\sin(\pi t)} dt$$
$$= \frac{(-1)^{k-1}(2\pi)^{2k}}{2\,(2k)!} B_{2k} \qquad (14)$$

and *in the odd case*

$$\zeta(2k+1) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^{2k+1}}$$
$$= \frac{(-1)^{k-1}(2\pi)^{2k+1}}{(2k+1)!} \lim_{n\to\infty} \int_0^1 B_{2k+1}(t) \frac{\cos(\pi t) - \cos((2n+1)\pi t)}{2\sin(\pi t)} dt$$
$$= \frac{(-1)^{k-1}(2\pi)^{2k+1}}{2\,(2k+1)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt. \qquad (15)$$

8.2 Class of functions $\mathscr{B}$

In mathematics, the most common expansions of functions are power series expansions, i.e., with respect to $\{(z-c)^n\}_{n=0}^{\infty}$ or Fourier series expansions, i.e., with respect to $\{\cos(nz)\}_{n=0}^{\infty}$ and $\{\sin(nz)\}_{n=1}^{\infty}$, namely

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n \quad \text{or} \quad f(z) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nz) + b_n \sin(nz)).$$

However, in many situations, expansions with respect to other families of functions are important as well. Taking into account the formulas (14) and (15) for values of the zeta function in terms of integrals, expansions of functions with respect to the Bernoulli polynomials seem to be useful. Not all functions can be expanded in terms of Bernoulli polynomials. There are some restrictions on the class of such functions. A very nice book [7] related to this subject is by Boas and Buck.

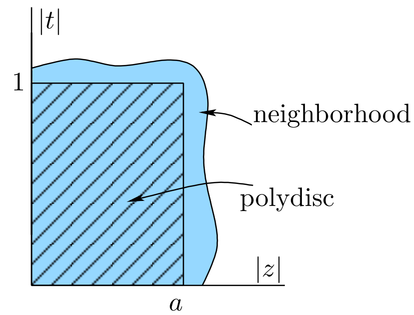


Figure 9: *Polydisc and its neighborhood*

We define the following class $\mathscr{B}$ of functions. Let $f = f(z,t)$ be a holomorphic function of two variables in a neighborhood of the closed polydisc, i.e., $|z| \leq a$ $(a > 0)$, $|t| \leq 1$,

$$f(z,t) = \sum_{n,k=0}^{\infty} a_{nk} z^n t^k,$$

for $|z| < a + \varepsilon$, $|t| < 1 + \varepsilon$, where $\varepsilon > 0$ is a small real number. Moreover, assume that f has the property

$$f(z,t) = \sum_{n=0}^{\infty} f_n(t) z^n,$$

and for each $n = 0, 1, \dots$ there exist constants c_{nj}, $j = 0, 1, \dots$, such that

$$f_n(t) = \sum_{j=0}^{\infty} c_{nj} B_j(t) \qquad \text{converges for} \quad |t| < 1 + \varepsilon,$$

that is, each f_n can be expanded into a series with respect to Bernoulli polynomials.

We also define a subclass $\mathscr{B}_0 \subset \mathscr{B}$ of functions. Let $f = f(z,t)$ be a holomorphic function of two variables in the closed polydisc $|z| \leq a$ $(a > 0)$, $|t| \leq 1$, as above. Moreover, assume that f has the property

$$f(z,t) = \sum_{n=0}^{\infty} f_n(t) z^n, \tag{16}$$

where $f_n(t)$ is a polynomial of degree l_n. Since

$$t^n = \frac{1}{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} B_k(t),$$

therefore we have $\mathscr{B}_0 \subset \mathscr{B}$.

8.3 Formulation of results

Let

$$f(z,t) = \sum_{n=0}^{\infty} f_n(t) z^n, \qquad f_n(t) = \sum_{j=0}^{\infty} c_{nj} B_j(t)$$

have the meaning as in the preceding subsection.

Theorem 4 ([10]) *If $f \in \mathscr{B}$ and additionally $f(z,0) = f(z,1) \equiv 0$, then*

$$\int_0^1 f(z,t)\cot(\pi t)\,dt = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{\infty} a_{n,2k+1}\,\zeta(2k+1) \right] z^n,$$

where

$$a_{n,2k+1} = \frac{2(-1)^{k+1}(2k+1)!}{(2\pi)^{2k+1}} c_{n,2k+1} \ .$$

Remark. The results are valid for any $f \in \mathscr{B}$ without the assumption $f(z,0) = f(z,1) \equiv 0$ if instead of the function $f = f(z,t)$ we take

$$\tilde{f}(z,t) = f(z,t) - f(z,0) - t(f(z,1) - f(z,0)).$$

Corollary 1 ([10]) *If $f \in \mathscr{B}_0$ and additionally $f(z,0) = f(z,1) \equiv 0$, then for each n there are complex constants $a_{n,2k+1}$, $3 \le 2k+1 \le l_n$, where $l_n = \deg\, f_n$, such that*

$$\int_0^1 f(z,t)\cot(\pi t)\,dt = \sum_{n=0}^{\infty} \left[\sum_{3 \le 2k+1 \le l_n} a_{n,2k+1}\zeta(2k+1) \right] z^n \ .$$

A natural question is: which functions $f(z,t)$ produce trivial relations in Theorem 4 and Corollary 1? In the cases

- $f = f(z,t)$ polynomial in $(z,\, t)$;
- $f = f(z,t) = g(z)h(t)$, where g, h are holomorphic functions;

then theorem 4 and Corollary 1 do not provide any information, just trivial identities.

However, if

- $f(z,t) = f(zt)$, where f is a holomorphic function in the closed unit disc (here f has a double meaning) with a power series with infinitely many nonzero coefficients, i.e.,

$$f(z,t) = \sum_{n=0}^{\infty} a_n t^n z^n, \quad a_n \neq 0 \quad \text{for infinitely many } n; \tag{17}$$

- a generalization of the above case: let $f = f(z,t)$ be from the class $\mathscr{B}_0$, i.e., as in (16), where $f_n(t)$ are polynomials; assume additionally that the degrees of these polynomials are not bounded, i.e., $\max_n(\deg f_n) = \infty$;
- most of the functions from the class $\mathscr{B}$;

we get nontrivial equalities.

Finally we formulate one more corollary of Theorem 4 when applying it to functions from (17).

Corollary 2 *Assume that $f(z,t) = f(zt)$ as in (17) and additionally $f(0) = 0$. Then*

$$\begin{aligned} &\int_0^1 [f(zt) - tf(z)] \cot(\pi t)\, dt = \\ &= \sum_{n=3}^{\infty} a_n \left[\sum_{3 \le 2k+1 \le n} \frac{2(-1)^{k+1}(2k+1)!}{(2\pi)^{2k+1}} \binom{n+1}{2k+1} \zeta(2k+1) \right] z^n \,. \end{aligned} \tag{18}$$

Some comments regarding the above corollary. If we can find a function f as in (17) and are able to evaluate explicitly the integral on the left-hand side of (18), say

$$\int_0^1 [f(zt) - tf(z)] \cot(\pi t)\, dt = \sum_{n=3}^{\infty} c_n z^n,$$

then from (18) we get finite relations between the values $\zeta(2k+1)$, namely

$$\sum_{3\le 2k+1\le n} \frac{2(-1)^{k+1}(2k+1)!}{(2\pi)^{2k+1}} \binom{n+1}{2k+1} \zeta(2k+1) = c_n/a_n \quad \text{if } a_n \neq 0.$$

This would be a fantastic result. However, the question whether it is possible or not to find such a function f is open. Similar considerations hold for any function from the class $\mathscr{B}_0$.

8.4 Applications

As an application of the results formulated above, we take the function

$$f(z,t) = -\frac{\pi z e^{2\pi i z t}}{e^{2\pi i z} - 1}\,,$$

which can be obtained from the definition of Bernoulli polynomials in (12), which we recall here,

$$\frac{z e^{zt}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(t) \frac{z^k}{k!}$$

by plugging $2\pi i z$ for z in the above equation and dividing the result by $(-2i)$. Applying Corollary 1 and after some calculations (see [10]), the following formulas hold:

$$\begin{aligned}
\sum_{k=1}^{\infty} \zeta(2k+1) z^{2k} &= -\frac{\pi}{e^{2\pi i z} - 1} \int_0^1 [e^{2\pi i z t} - 1 - t(e^{2\pi i z} - 1)] \cot(\pi t)\, dt \\
&= \frac{2z}{\sin(2\pi z)} \int_0^{\pi} \cos(2zt) \ln(\sin t)\, dt \;+\; \ln 2 \\
&= \frac{2z}{1 - \cos(2\pi z)} \int_0^{\pi} \sin(2zt) \ln(\sin t)\, dt \;+\; \ln 2.
\end{aligned}$$

One more application to the digamma function ψ, which is defined

$$\psi(z) := \frac{d}{dz}[\ln \Gamma(z)] = \frac{\Gamma'(z)}{\Gamma(z)}.$$

In the literature the following formula is known

$$-\frac{1}{2}[2\gamma + \psi(z) + \psi(-z)] = \sum_{k=1}^{\infty} \zeta(2k+1) z^{2k}, \tag{19}$$

where γ is the Euler constant. We note that the right-hand side in (19) coincides with the previous evaluation above. Consequently we get the identity

$$-\frac{1}{2}[2\gamma + \psi(z) + \psi(-z)] = \frac{2z}{\sin(2\pi z)} \int_0^\pi \cos(2zt) \ln(\sin t)\, dt + \ln 2.$$

Acknowledgments

Roman J. Dwilewicz is partially supported by the internal grant of the Cardinal Stefan Wyszyński University, Warsaw, Poland, 2009, and Ján Mináč is partially supported by the Natural Sciences and Engineering Council of Canada grant R0370A02.

References

[1] M. Abramowitz and I.A. Stegun (eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications Inc., New York 1972.

[2] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$. Astérisque **61** (1979), 11–13.

[3] T.M. Apostol, *Introduction to Analytic Number Theory.* Springer - Verlag, New York - Heidelberg - Berlin 1976.

[4] K. Ball and T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zeta aux entiers impairs.* Invent. Math. **146** (2001), no. 1, 193–207.

[5] H. Bateman (A. Erdelyi ed.), *Higher Transcendental Functions.* Vol. 1, McGraw-Hill Book Company 1953.

[6] B.C. Berndt, *Elementary evaluation of* $\zeta(2n)$. Mathematics Magazine **48** (1975), 148–154.

[7] R.P. Boas and R.C. Buck, *Polynomial Expansions of Analytic Functions.* Springer - Verlag, Berlin - Göttingen - Heidelberg 1964.

[8] J. Bruna, *Euler, sèries i funció zeta de Riemann.* Preprint (2007).

[9] K. Dilcher, L. Skula, and I.S. Slavutskii, *Bernoulli Numbers. Bibliography (1713–1990).* Queen's Papers in Pure and Applied Mathematics, 87. Queen's University, Kingston, ON, 1991. iv+175 pp.

[10] R. Dwilewicz and J. Mináč, *An introduction to relations between the values of $\zeta(s)$ in terms of holomorphic functions of two variables.* Proceedings of the *Hayama Symposium on Several Complex Variables,* Japan, Dec. 2000. Pages 28–38 (2001).

[11] R. Dwilewicz and J. Mináč, *The Hurwitz zeta function as a convergent series.* Rocky Mountain J. Math. **36** (2006), 1191–1219.

[12] G. Everest, C. Röttger and T. Ward, *The continuing story of zeta.* The Math. Intelligencer **31** (2009), 13–17.

[13] A.B. Goncharov, *The classical polylogarithms, algebraic K-theory and $\zeta_F(n)$.* The Gel'fand Mathematical Seminars, 1990 - 1992, Birkhäuser Boston, Boston, MA, 1993, 113–135.

[14] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory.* Springer - Verlag (GTM), vol. 84 (1990).

[15] E. Landau, Differential and Integral Calculus. Chelsea Publishing Company 1960.

[16] A.I. Markushevich, *Theory of Functions of a Complex Variable.* Vol. I, II, III. Prentice-Hall, Inc., 1965.

[17] J. Mináč, *A remark on the values of the Riemann zeta function.* Exposition. Math. **12** (1994), no. 5, 459–462.

[18] M.R. Murty and M. Reece, *A simple derivation of $\zeta(1-K) = -B_K/K$.* Funct. Approx. Comment. Math. **28** (2000), 141–154.

[19] T. Rivoal, *La fonction zeta de Riemann prend une infinité de valeurs irrationnelles aux entier impairs.* C.R. Acad. Sci. Paris Sér. I Math. **331** (2000), 267–270.

[20] T. Rivoal, *Irrationalité d'au moins un des neuf nombres $\zeta(5)$, $\zeta(7)$, $\ldots, \zeta(21)$.* Acta Arith. **103** (2002), no. 2, 157–167.

[21] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function.* Claredon Press, Oxford 1986.

[22] A. Van der Poorten, *A proof that Euler missed...Apéry's proof of the irrationality of* $\zeta(3)$*. An informal report.* Math. Intelligencer **1** (1978/79), no. 4, 195–203.

[23] A. Weil, *Number Theory. An Approach Through History From Hammurapi to Legendre.* Birkhäuser, Boston 1984.

[24] V.V. Zudilin, *One of the numbers* $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ *is irrational.* (Russian) Uspekhi Mat. Nauk **56** (2001), no. 4(340), 149–150; translation in Russian Math. Surveys **56** (2001), no. 4, 774–776.

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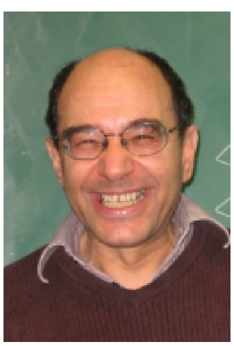

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Publicat el 10 de novembre de 2009