Rational homotopy of non-connected spaces

Urtzi Buijs
(Joint work with Aniceto Murillo)

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Université de Valenciennes et du Hainaut-Cambrésis

1. A brief introduction

Problem: How to realize algebraic structures?

$$\mathbf{CDGA} \overset{\langle\,\cdot\,\rangle}{\rightsquigarrow} \mathbf{SSet}$$

Category of simplicial sets

Category of commutative differential graded (over $\mathbb{Z}$) algebras over a field of characteristic zero

$$\langle A\rangle_n = Hom\big(A, (A_{PL})_n\big) \qquad (A_{PL})_n = \Lambda(t_1,\ldots,t_n,dt_1,\ldots,dt_n)$$

Sullivan, Publications Mathématiques de l'IHÉS 1977
Bousfield, Guggenheim, Mem. Amer. Math. Soc. 1976

Theorem

With the appropriate restrictions, the induced functor in the homotopy categories is an equivalence.

$$Ho\,\mathbf{CDGA}^{\mathbb{Q}}_{\mathcal{N},f} \overset{\cong}{\rightsquigarrow} Ho\,\mathbf{SSet}^{\mathbb{Q}}_{\mathcal{N},f}$$

1. A brief introduction

$$\mathrm{DGL} \underset{\lambda}{\overset{\langle\,\cdot\,\rangle}{\rightsquigarrow}} \mathrm{SSet}$$

Category of differential graded (over $\mathbb{Z}$) Lie algebras over a field of characteristic zero

Theorem

Quillen, Annals of Mathematics, 1969

Neisendorfer, Pacific J. of Math. 1978

With the appropriate restrictions, the induced functors in the homotopy categories are equivalences.

$$Ho\,\mathrm{DGL}_1^{\mathbb{Q}} \overset{\cong}{\rightsquigarrow} Ho\,\mathrm{SSet}_1^{\mathbb{Q}}$$

$$Ho\,\mathrm{DGL}_{\mathcal{N},f}^{\mathbb{Q}} \overset{\cong}{\rightsquigarrow} Ho\,\mathrm{SSet}_{\mathcal{N},f}^{\mathbb{Q}}$$

The *"unrestricted"* situation is also useful...

1. A brief introduction

Let X be a finite nilpotent CW-complex.

Let Y be a finite type nilpotent CW-complex.

Let C be a finite dimensional coalgebra model of X.

Let L be a DGL model of Y. el of Y.

Haefliger, Trans. of the Amer. Math. Soc. 1982
Brown, Szczarba, Trans. of the Amer. Math. Soc. 1997

Theorem $\big(\Lambda(V \otimes C), D\big)$ is a *model* of $\mathrm{map}(X,Y)$.

Theorem $Hom(C,L)$ is a DGL *model* of $\mathrm{map}(X,Y)$.

Scherer, Tanré, Arch. Math. 1992
B. , Félix, Murillo, Trans. of the Amer. Math. Soc. 2009

$$[g,h] : C \longrightarrow L,$$

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{g \otimes h} L \otimes L \xrightarrow{[\,,\,]} L$$

1. A brief introduction

Goal

Develop a consistent *homotopy theory* in (unbounded!) $\mathbf{DGL}$, or more generally in $\mathbf{L}_\infty$, to be able to algebraically model the homotopy type of non-connected spaces.

2. Cochain functor

Definition

An L_∞ algebra structure on a graded vector space L is a family of linear maps of degree $k-2$

$$\ell_k\colon \otimes^k L \to L$$

satifying:

(1) Graded skew-symmetry.

(2) Higher Jacobi identities.

Equivalently $(\Lambda sL, \delta)$

$\mathcal{C}_*(L) = (\Lambda sL, \delta)$ is the *Cartan-Eilenberg-Chevalley* CDGC.

A morphism of L_∞ algebras $\phi\colon L \to L'$ is morphism of CDGC

$$(\Lambda sL, \delta) \to (\Lambda sL', \delta')$$

Equivalently, it is a family of linear maps $\phi^{(k)}\colon L^{\otimes k} \to L'$ of degree $k-1$ satisfying some relations involving $\{\ell_k\}$ and $\{\ell'_k\}$.

2. Cochain functor

Definition

An L_∞ algebra L is *mild* if every bracket is locally finite, i.e., for any $a \in L$ there are finite dimensional subspaces $S_k \subset \otimes^k L$, $k \geq 1$ which vanish for $k \gg 0$ and such that

$$\ell_k^{-1}\langle a \rangle \subset \mathrm{Ker}\,\ell_k \oplus S_k.$$

Definition

Given a mild L_∞ algebra L, choose a homogeneous basis $\{z_i\}$ of L and denote by $V \subset (sL)^\sharp$, $V = \langle \{v_i\} \rangle$ where $v_i(sz_r) = \delta_i^r$.

$$\mathcal{C}^*(L) = (\Lambda V, d), \quad d = \sum_{k \geq 1} d_k, \; d_k V \subset \Lambda^k V$$

$$\langle d_k v; sx_1 \wedge \cdots \wedge sx_k \rangle = \pm \langle v; s\ell_k(x_1, \ldots, x_k) \rangle.$$

2. Cochain functor

$$\langle d_k v; sx_1 \wedge \cdots \wedge sx_k \rangle = \pm \langle v; s\ell_k(x_1, \ldots, x_k) \rangle.$$

L is mild if for each v_i

$\langle v; s\ell_k(z_{j_1}, \ldots, z_{j_k}) \rangle = 0$

for almost all $z_{j_1} \otimes \cdots \otimes z_{j_k} \in L^{\otimes k}$

$(\Lambda V, d)$ is mild if for each $z_{j_1} \otimes \cdots \otimes z_{j_k} \in L^{\otimes k}$

$\langle d_k v_i; sz_{j_1} \wedge \cdots \wedge sz_{j_k} \rangle = 0$

for almost all $v_i \in V$.

2. Cochain functor

$\mathcal{C}^*(-)$ does not define a functor unless we also restrict the class of L_∞ morphisms.

$\phi\colon (\Lambda sL, \delta) \to (\Lambda sM, \delta)$ is *mild* if every $\phi^{(k)}\colon \Lambda^k sL \to sM$ is locally finite, i.e. for any $a \in M$ there is a finite dimensional subspace $S_k \subset \otimes^k L$, $k \geq 1$, with $S_k = 0$ $k \gg 0$, such that

$$(\phi^{(k)})^{-1}\langle a\rangle = \mathrm{Ker}\phi^{(k)} \oplus S_k.$$

If $\phi\colon (\Lambda sL, \delta) \to (\Lambda sM, \delta)$ is mild, define $\mathcal{C}^*(\phi)\colon \mathcal{C}^*(M) = (\Lambda W, d) \to (\Lambda V, d) = \mathcal{C}^*(L)$, with $\mathcal{C}^*(\phi) = \sum_{k\geq 1} \mathcal{C}^*(\phi)_k$ where $\mathcal{C}^*(\phi)_k\colon W \to \Lambda^k V$ is given by

$$\langle \mathcal{C}^*(\phi)_k w; sx_1 \wedge \cdots \wedge sx_k\rangle = \pm\langle w; s\phi^{(k)}(x_1 \otimes \cdots \otimes x_k)\rangle.$$

We will denote this category by $\mathbb{L}_\infty^{\mathrm{mild}}$.

2. Cochain functor

$$\langle \mathcal{C}^*(\phi)_k w; sx_1 \wedge \cdots \wedge sx_k \rangle = \pm \langle w; s\phi^{(k)}(x_1 \otimes \cdots \otimes x_k) \rangle.$$

Let L and M be mild L_∞ algebras
$\phi\colon L \to M$ is mild if for each w_i

$$\langle w_i; s\phi^{(k)}(z_{j_1} \otimes \cdots \otimes z_{j_k}) \rangle = 0$$

for almost all $z_{j_1} \otimes \cdots \otimes z_{j_k} \in \otimes^k L$.

Let $(\Lambda V, d)$ and $(\Lambda W, d)$ be mild CDGA's.
$\psi\colon (\Lambda W, d) \to (\Lambda V, d)$ is mild
if for each $z_{j_1} \otimes \cdots \otimes z_{j_k} \in \otimes^k L$

$$\langle \psi_k w_i; sz_{j_1} \wedge \cdots \wedge sz_{j_k} \rangle = 0$$

for almost all w_i.

2. Cochain functor

Remark 1

$$\textbf{FINITE TYPE + BOUNDED} \neq \textbf{MILD}$$

Example 1

Consider the L_∞ algebra

$$L = L_{-1} \oplus L_{-2}, \quad \text{where} \quad L_{-1} = \langle a \rangle \ ; \ L_{-2} = \langle b \rangle.$$

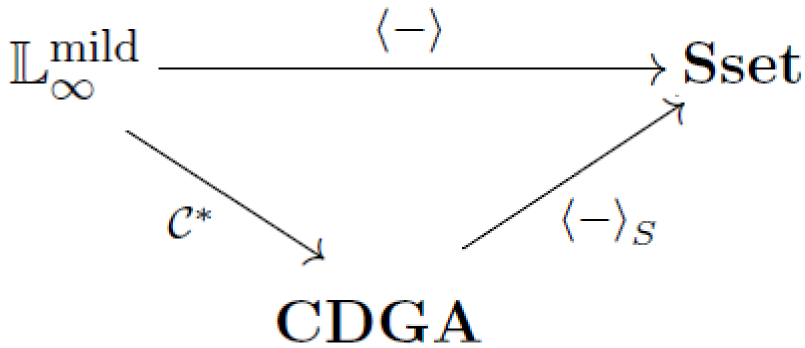
Brackets given by $\ell_k(a, \ldots, a) = b, \ k \geq 1$

L is NOT mild but is of finite type and bounded.

Example 2

Any non-finite type, non-bounded abelian L_∞ algebra ($\ell_k = 0, \ k \geq 1$) is trivially mild.

2. Cochain functor

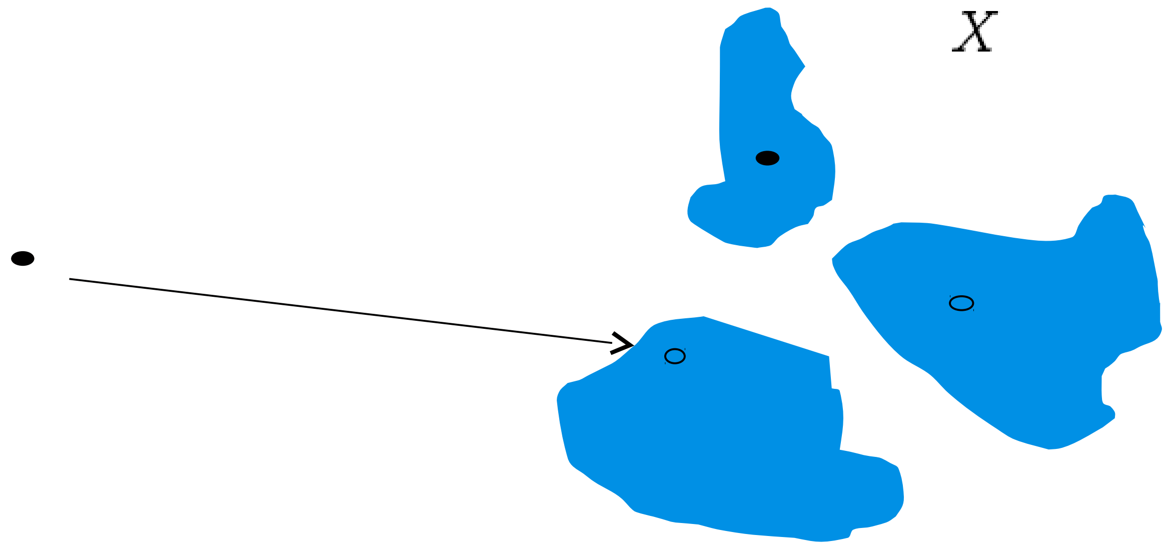


3. Points, augmentations and Maurer-Cartan elements.

$(\Lambda V, d) \in \mathsf{CDGA}$ $\qquad \langle \Lambda V, d\rangle_S \simeq X,$ non-connected space.

$$\varphi \in \langle(\Lambda V, d)\rangle_0 = Hom((\Lambda V, d), (A_{PL})_0) = Hom((\Lambda V, d), \mathbb{Q})$$

$$* \hookrightarrow X$$

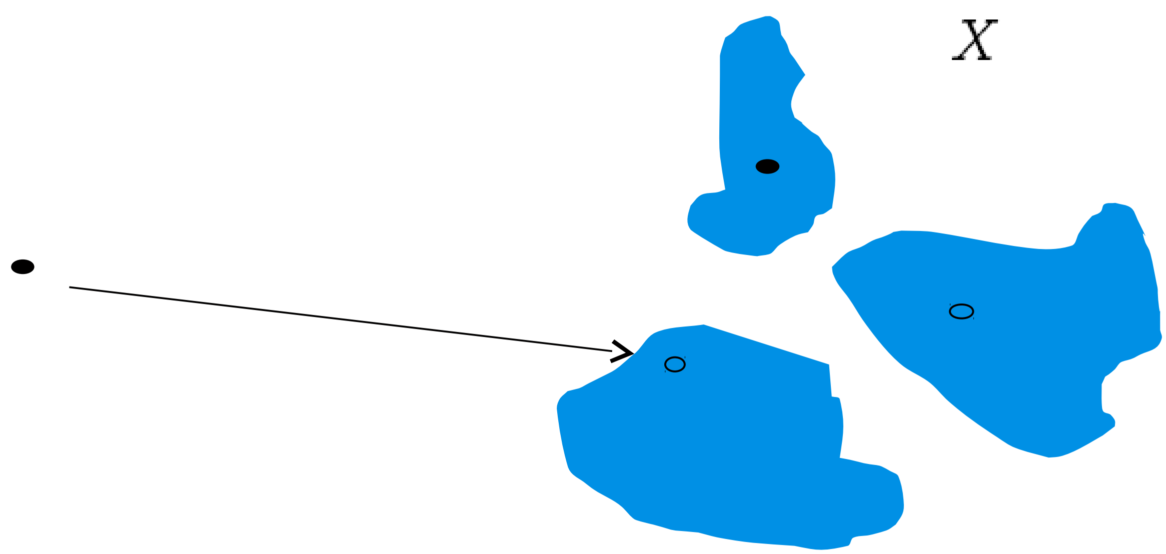


3. Points, augmentations and Maurer-Cartan elements.

$$K_\varphi = \{v \in V^{<0}\} \cup \{dv \ : \ v \in V^0\} \cup \{v - \varphi(v) \ : \ v \in V^0\}$$

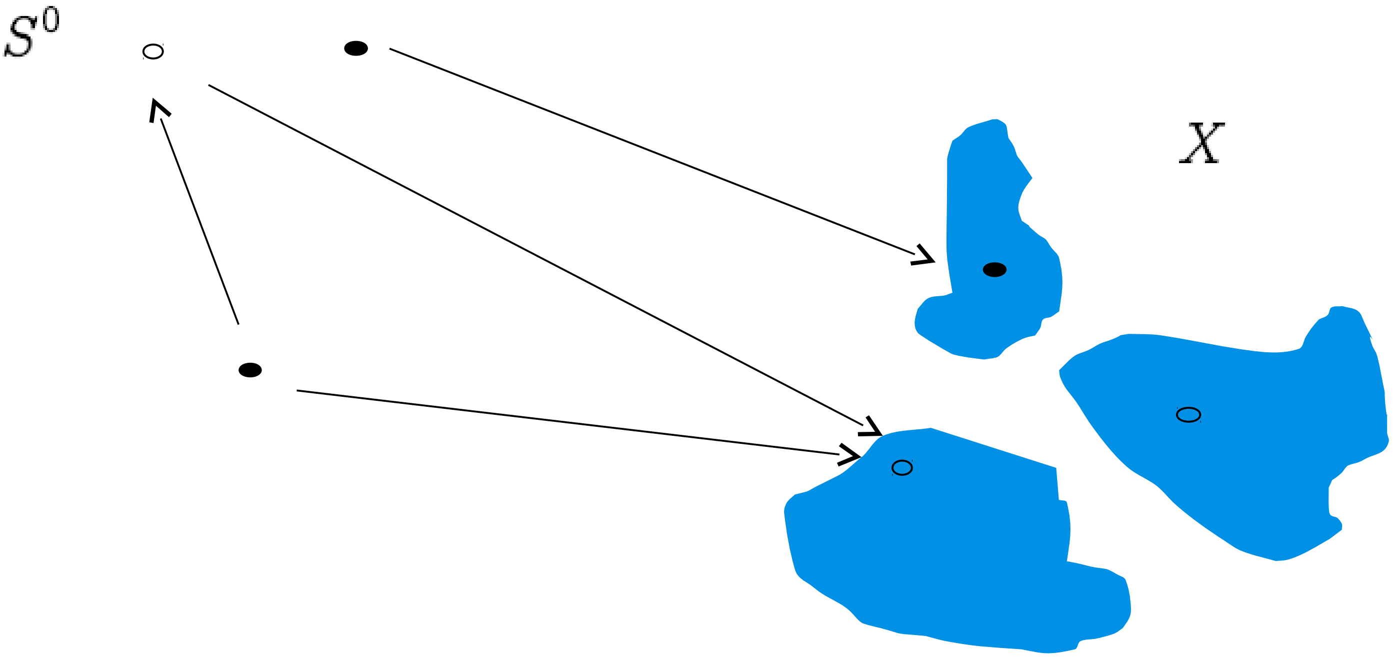
$$\langle (\Lambda V, d)/K_\varphi \rangle \simeq \langle (\Lambda V, d) \rangle_\varphi \simeq \mathrm{Comp}(X, x_0).$$

$$(\Lambda V, d)/K_\varphi \cong (\Lambda \overline{V}^1 \oplus V^{\geq 2}, d_\varphi).$$



3. Points, augmentations and Maurer-Cartan elements.

What about the DGL/L_∞ setting?



3. Points, augmentations and Maurer-Cartan elements.

What is a good model for S^0?

$$(\mathbb{L}(u), \partial), \quad |u| = -1 \;\; \partial(u) = -\frac{1}{2}[u,u] \qquad \mathbb{L}(u) = u \oplus [u,u].$$

$$\mathcal{C}^*(\mathbb{L}(u)) = (\Lambda(x,y), d), \quad |x| = 0, |y| = -1, dx = 0, dy = \frac{1}{2}(x^2 - x)$$

$$\langle \mathcal{C}^*(\mathbb{L}(u)) \rangle \simeq S^0.$$

Definition

Let L be an L_∞ algebra. $z \in L_{-1}$ is a *Maurer-Cartan element* if $\ell_k(z, \dots, z) = 0$ for $k \gg 0$ and

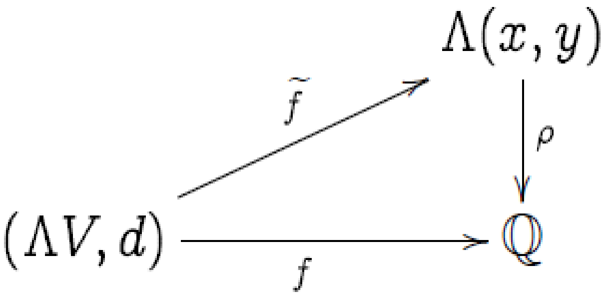
$$\sum_{k \geq 1} \frac{1}{k!} \ell_k(z, \dots, z) = 0.$$

3. Points, augmentations and Maurer-Cartan elements.

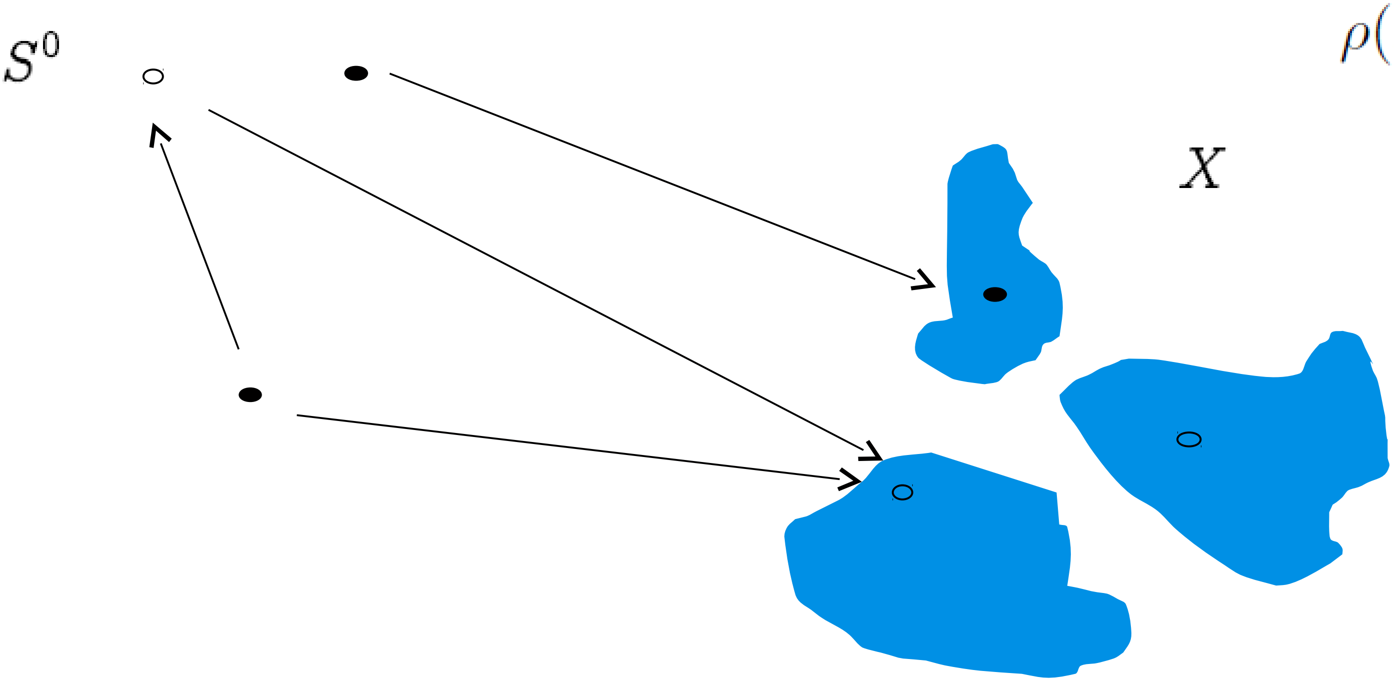
Lemma

Let $f\colon \Lambda V \to \mathbb{Q}$ be an augmentation.

There is a unique morphism of CGDA's $\widetilde{f}\colon \Lambda V \to \Lambda(x,y)$, linear in x, making the following diagram commutative.



$\rho(x) = 1$



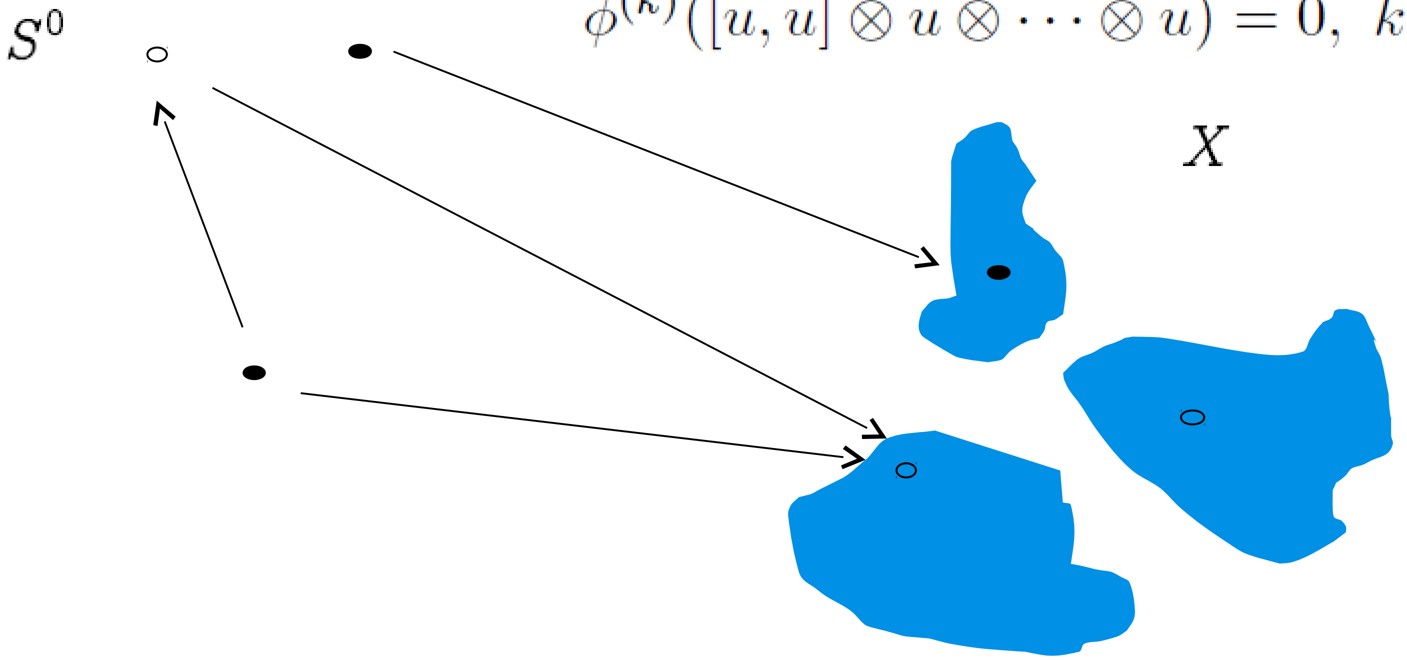
3. Points, augmentations and Maurer-Cartan elements.

Lemma

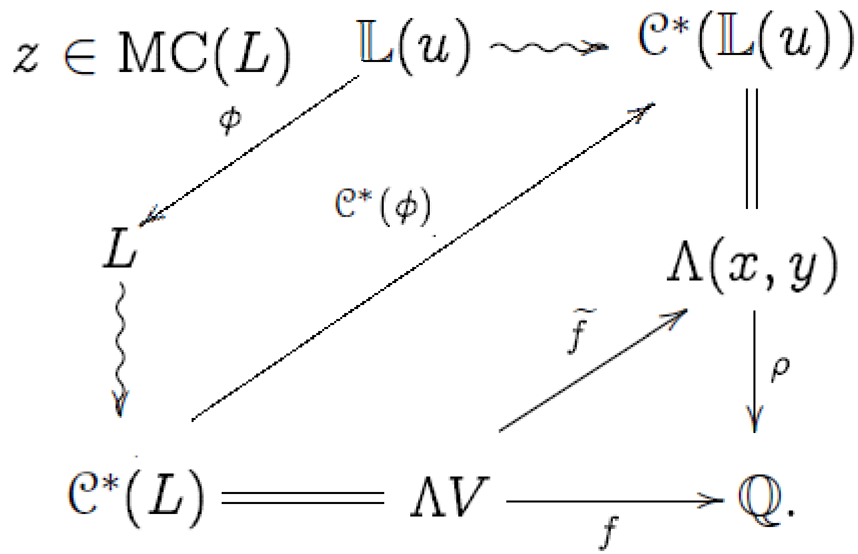
Let L be an L_∞ algebra. For any $z \in L_{-1}$ there exist a unique L_∞ morphism $\phi\colon \mathbb{L}(u) \to L$ such that

(1) $\phi^{(1)}(u) = z$, (2) $\phi^{(k)}(u \otimes \cdots \otimes u) = 0, k \geq 2.$

Moreover, $z \in \mathrm{MC}(L)$ if and only if $\phi^{(k)}([u,u] \otimes u \otimes \cdots \otimes u) = 0,\ k \gg 0.$



3. Points, augmentations and Maurer-Cartan elements.



3. Points, augmentations and Maurer-Cartan elements.

Lemma

If L is a mild L_∞ algebra of finite type, then: $z \in L_{-1}$ is a Maurer-Cartan element if and only if there exists a mild L_∞ morphism $\phi\colon \mathbb{L}(u) \to L$ such that $\phi^{(1)}(u) = z$ and $\phi^{(k)}(u \otimes \cdots \otimes u) = 0$ for $k \geq 2$.

From now on we will consider the category of finite type mild L_∞ algebras, denoted by $\mathbb{L}_\infty^{\text{mild, f.t.}}$.

3. Points, augmentations and Maurer-Cartan elements.

Theorem

(1) If $L \in \mathbb{L}_\infty^{\text{mild, f.t.}}$ then $\mathrm{MC}(L) \cong \mathrm{Aug}(\mathcal{C}^*(L))$.

(2) If $g\colon M \to L$ is a morphism in $\mathbb{L}_\infty^{\text{mild, f.t.}}$ then the map

$$\mathrm{MC}(g)\colon \mathrm{MC}(M) \to \mathrm{MC}(L)$$

$$z \mapsto \mathrm{MC}(g)(z) = \sum_{k\geq 1} \frac{1}{k!} g^{(k)}(z \otimes \cdots \otimes z)$$

is well defined.

(3) $\mathrm{MC}\colon \mathbb{L}_\infty^{\text{mild, f.t.}} \to \mathbf{Set}$ is naturally equivalent to $\mathrm{Aug}\,\mathcal{C}^*\colon \mathbb{L}_\infty^{\text{mild, f.t.}} \to \mathbf{Set}$.

3. Points, augmentations and Maurer-Cartan elements.

$$L \mapsto \mathrm{MC}(L)$$

$$\phi\colon M \to L$$

$$\downarrow$$

$$\mathrm{MC}(\phi)\colon \mathrm{MC}(M) \to \mathrm{MC}(L)$$

$$\mathrm{MC}(\phi)(u) = \sum_{k\geq 0} \frac{1}{k!}\phi^{(k)}(u \otimes \cdots \otimes u)$$

$$L$$

$$\downarrow$$

$$\mathrm{Aug}\,\mathcal{C}^*(L) = \{f\colon \mathcal{C}^*(L) = (\Lambda V, d) \to \mathbb{Q}\}$$

$$\phi\colon M \to L$$

$$\downarrow$$

$$\mathrm{Aug}\,\mathcal{C}^*(\phi)\colon \mathrm{Aug}\,\mathcal{C}^*(M) \to \mathrm{Aug}\,\mathcal{C}^*(L)$$

$$\mathcal{C}^*(M) \xleftarrow{\mathcal{C}^*(\phi)} \mathcal{C}^*(L), \qquad f\colon \mathcal{C}^*(M) \to \mathbb{Q}, \qquad f\circ\mathcal{C}^*(\phi)\colon \mathcal{C}^*(L) \to \mathbb{Q}$$

(3) $\mathrm{MC}\colon \mathbb{L}_\infty^{\text{mild, f.t.}} \to \mathbf{Set}$ is naturally equivalent to $\mathrm{Aug}\,\mathcal{C}^*\colon \mathbb{L}_\infty^{\text{mild, f.t.}} \to \mathbf{Set}$.

3. Points, augmentations and Maurer-Cartan elements.

Definition
Given an L_∞ algebra L and $z \in \mathrm{MC}(L)$, the *perturbation* of ℓ_k by z is defined by

$$\ell_k^z(x_1, \dots, x_k) = \sum_{i \geq 0} \frac{1}{i!} \ell_{i+k}(z, \overset{i}{\dots}, z, x_1, \dots, x_k).$$

$(L, \{\ell_i^z\})$ is again an L_∞ algebra denoted by L^z.

If L is mild then L^z is also mild.

$$(L^{(z)})_i = \begin{cases} L_i & \text{if } i > 0, \\ \mathrm{Ker}\,\ell_1^z & \text{if } i = 0 \\ 0 & \text{if } i < 0 \end{cases}$$

3. Points, augmentations and Maurer-Cartan elements.

Theorem

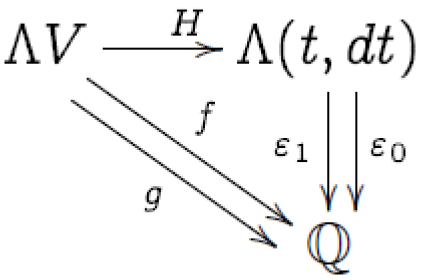
Let $L \in \mathbb{L}_\infty^{\text{mild, f.t.}}$ and let $f : \mathcal{C}^*(L) \to \mathbb{Q}$ be an augmentation corresponding to a Murer-Cartan element $z \in \mathrm{MC}(L)$. Then

$$\langle \mathcal{C}^*(L) \rangle_f \simeq \langle \mathcal{C}^*(L^{(z)}) \rangle.$$

Proof: $\mathcal{C}^*(L^{(z)}) \cong (\Lambda V, d)/K_f.$

4. Lawrence-Sullivan interval and homotopy

What can we say if two augmentations are homotopic $f \simeq g$?



How can we translate this homotopy in terms of L_∞ algebras and Maurer-Cartan elements z_f and z_g?

In that case what is the relation between $L^{(z_f)}$ and $L^{(z_g)}$?

4. Lawrence-Sullivan interval and homotopy

To do that we will use a particular object called the Lawrence-Sullivan interval

$$\widehat{\mathbb{L}}(a,b,x), \quad |a|=|b|=-1, \ |x|=0$$

$$\partial(a) = -\frac{1}{2}[a,a], \ \partial(b) = -\frac{1}{2}[b,b]$$

$$\partial(x) = \mathrm{ad}_x(b) + \sum_{i\geq 0} \frac{B_i}{i!}\mathrm{ad}_x^i(b-a),$$

$$B_0 = 1, \quad \frac{B_n}{n!} = -\sum_{k=0}^{n-1} \frac{B_k}{k!}\frac{1}{(n+1-k)!} \quad \text{para } n \geq 1.$$

Theorem

R. Lawrence, D. Sullivan

$$\partial^2 = 0.$$

4. Lawrence-Sullivan interval and homotopy

Definition

Let $L \in L \in \mathbb{L}_\infty^{\text{mild, f.t.}}$. Let $z, z' \in \mathrm{MC}(L)$.
We say that $z \simeq z'$ if there is an L_∞ morphism

$$\phi\colon \widehat{\mathbb{L}}(a, b, x) \to L$$

such that $\mathrm{MC}(\phi)(a) = z$ and $\mathrm{MC}(\phi)(b) = z'$.

Theorem

Let $L \in \mathbb{L}_\infty^{\text{mild, f.t.}}$ and $z_f, z_g \in \mathrm{MC}(L)$ be the corresponding Maurer-Cartan elements associated to the augmentations $f, g\colon \mathcal{C}^*(L) \to \mathbb{Q}$. Then

$$f \simeq_H g \quad \text{if and only if} \quad z_f \simeq z_g.$$

Proof:

$$\circlearrowleft \Lambda(t, dt) \leftrightarrows \langle \alpha, \beta, \gamma \rangle \overset{\mathcal{L}_\infty}{\rightsquigarrow} \widehat{\mathbb{L}}(s^{-1}\alpha^*, s^{-1}\beta^*, s^{-1}\gamma^*) = \widehat{\mathbb{L}}(a, b, x).$$

A_∞ algebra

4. Lawrence-Sullivan interval and homotopy

Theorem
Let $L \in \mathbb{L}_\infty^{\text{mild, f.t.}}$. Let $z, z' \in \mathrm{MC}(L)$ be two homotopic Maurer-Cartan elements $z \simeq z'$. Then

$$\langle L^{(z)} \rangle \simeq \langle L^{(z')} \rangle.$$

Proof:

$$(X, x_0) \xleftarrow{\simeq} X^I \xrightarrow{\simeq} (X, x_1)$$

$$(\Lambda V, d)/K_f \xrightarrow{\simeq} (\Lambda(V \otimes \langle \alpha^*, \beta^*, \gamma^* \rangle), \widetilde{d})/K_H \xleftarrow{\simeq} (\Lambda V, d)/K_g.$$

Then $$\langle (\Lambda V, d)/K_f \rangle \simeq \langle (\Lambda V, d)/K_g \rangle$$

$$\langle L^{(z)} \rangle = \langle (\Lambda V, d)/K_f \rangle \simeq \langle (\Lambda V, d)/K_g \rangle = \langle L^{(z')} \rangle.$$

4. Lawrence-Sullivan interval and homotopy

Definition $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\simeq .$

Then, given $g\colon L \to M$, a morphism in $\mathbb{L}_\infty^{\text{mild, f.t.}}$,

$\mathrm{MC}(g)$ induce a map $\widetilde{\mathrm{MC}}(g)\colon \widetilde{\mathrm{MC}}(L) \to \widetilde{\mathrm{MC}}(M)$.

Theorem
Let $g\colon L \to M$ be a morphism in $\mathbb{L}_\infty^{\text{mild, f.t.}}$ such that $\mathcal{C}^*(g)$ is a quasi-isomorphism of cofibrant CDGA's. Then

$$\widetilde{\mathrm{MC}}\colon \widetilde{\mathrm{MC}}(L) \xrightarrow{\cong} \widetilde{\mathrm{MC}}(M)$$

Proof:

$$\begin{array}{ccc} \mathcal{C}^*(L) & \xleftarrow[\simeq]{\mathcal{C}^*(g)} & \mathcal{C}^*(M) \\ & \searrow \quad \swarrow & \\ & \mathbb{Q} & \end{array}$$

4. Lawrence-Sullivan interval and homotopy

Theorem
Let $L \in \mathbb{L}_{\infty}^{\text{mild, f.t.}}$, then

$$\langle L \rangle \simeq \bigcup_{z \in \widetilde{\mathrm{MC}}(L)} \langle L^{(z)} \rangle .$$

B., Murillo, Adv. In Math 2013

5. Algebraic models of non-connected spaces

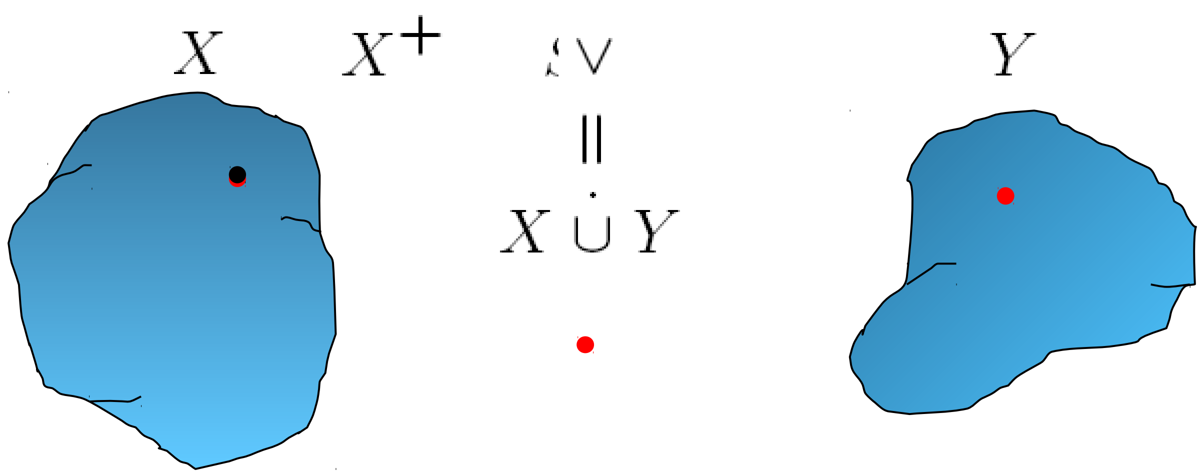
Input:

Family of nilpotent spaces of the homotopy type of finite type CW-complexes.

Family of 0-reduced DGL's of finite type.

Output:

DGL whose realization is of the homotopy type of the rationalization of the space whose components are the elements of the given family.



5. Algebraic models of non-connected spaces

Given $L, M \in \mathbf{DGL}$ we denote by $L * M$ its coproduct. Recall that, given free presentations $L = \mathbb{L}(U)/I$, $M = \mathbb{L}(V)/J$, then $L * M = \mathbb{L}(U \oplus V)/\langle I, J\rangle$

Let L, M be non-negatively graded DGL's models of the path connected spaces X and Y.

Lemma $\mathbb{L}(u) * L$ is a model of $X \vee S^0$.

Perturb the differential by the Maurer-Cartan element u:

$$(\mathbb{L}(u) * L, \partial_u)$$

$$\partial_u(u) = \frac{1}{2}[u, u]$$

$$\partial_u a = \partial a + [u, a], \; a \in L$$

Nothing has changed! (except the base point)

Indeed, $\langle L\rangle \simeq \langle L_z\rangle$ for any $z \in \mathrm{MC}(L)$.

Lemma $(\mathbb{L}(u) * L, \partial_u)$ is a model of $X^+ = X \,\dot{\cup}\, \{\text{point}\}$.

Finally,

Theorem $(\mathbb{L}(u) * L * M, \partial_u * \partial_M)$ is a model of $X \,\dot{\cup}\, Y$.

More generally,

Theorem

Let X be a space with path components $\{Y, X_j\}_{j\in J}$ and let $\{L, L_j\}_{j\in J}$ be a family of non-negatively graded DGL's, each of which modeling the corresponding component. For each $j \in J$ consider the perturbed DGL

$$M_j = (\mathbb{L}(u_j) * L_j, \partial_{u_j}),$$

$$\partial_{u_j}(u_j) = \frac{1}{2}[u_j, u_j], \qquad \partial_{u_j} x = \partial_j x + [u_j, x], \qquad x \in L_j.$$

Then,

$$M = *_{j\in J} M_j * L$$

is a model of X.

Merci!