Introduction to the Catastrophe Theory

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Barcelona

II Spanish Young Topologists Meeting

Part I: A bit of Mechanics

Lagrangian Systems

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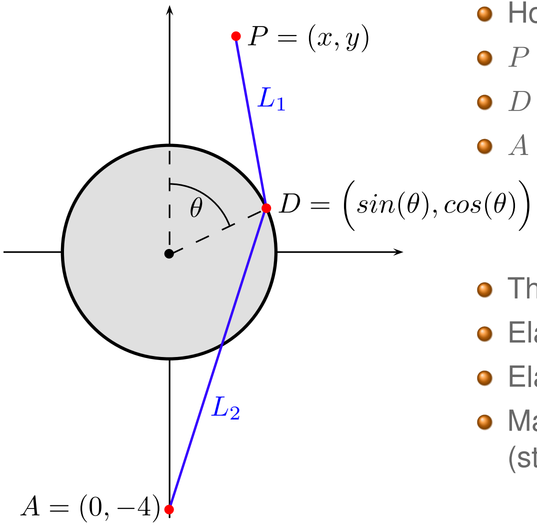
- We are interested in the statics and not in the dynamics.
- The extrema of the potential V gives the equilibria of the system.

Part II: Zeeman's Catastrophe Machine

Zeeman's Machine

Sir Erik Christopher Zeeman
1925 –

Zeeman's Machine



- Horizontal Plane.
- P is mobile.
- D is fixed to the disk.
- A is fixed to the plane.

- The disk can rotate freely.
- Elastic joining A with D.
- Elastic joining D with P.
- Masses are unimportant (static).

Zeeman's Machine

The potential energy is given by:

$$V = \frac{k(L_1 - 2)^2}{2} + \frac{k(L_2 - 2)^2}{2}$$

With generalized coordinates:

$$V_{(x,y)}(\theta) = \frac{k}{2}\left[\left(2 - \sqrt{(c\theta - x)^2 + (s\theta - y)^2}\right)^2 + \left(2 - \sqrt{(c\theta)^2 + (s\theta + 4)^2}\right)^2\right]$$

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- (x, y) can be considered as control parameters. $V_{(x,y)}$ biparametric family of potentials.
- For every (x, y) we look for the extrema of $V_{(x,y)}$, i.e. θ_0 such that $V'_{(x,y)}(\theta_0) = 0$.

Zeeman's Machine

Questions

Zeeman's Machine

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- Does this extremum exist?

Zeeman's Machine

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- Is it unique?

Zeeman's Machine

Questions

- Does this extremum exist?
- Is it unique?
- Do the answers depend on (x, y)?

Zeeman's Machine

Nonlinear dynamics, Drexel University: http://lagrange.physics.drexel.edu/flash/zcm/

Zeeman's Machine - Properties

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There are loops over the parameter space that lead smoothly to different equilibria.

This behaviour is known as **divergence**.

Zeeman's Machine - Simplification

V is hard to work with, but we can obtain an “alternative potential” (canonical form) $\overline{V}$, that closed to a degenerate critical point, behaves like V:

$$\overline{V}_{(x,y)}(\theta) = a_0\theta^4 + x\theta + y\theta^2$$

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$$\overline{V}_{(x,y)}(\theta) = a_0\theta^4 + x\theta + y\theta^2$$

Remark

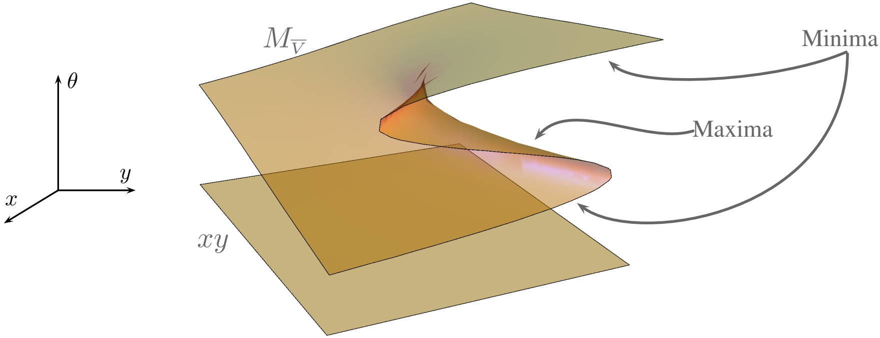
It is proven that the relevant information is preserved.

Zeeman's Machine - Study

Definition

The **surface of equilibria**:

$$M_{\overline{V}} = \{(x, y, \theta) \;/\; \overline{V}'_{(x,y)}(\theta) = 0\} = \{(x, y, \theta) \;/\; 4a_0\theta^3 + x + 2y\theta = 0\}$$



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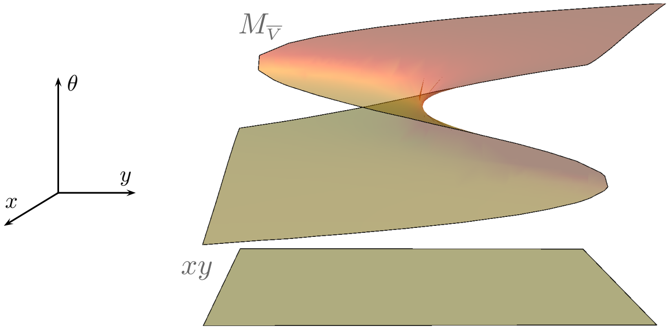
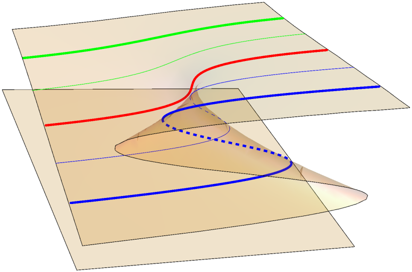


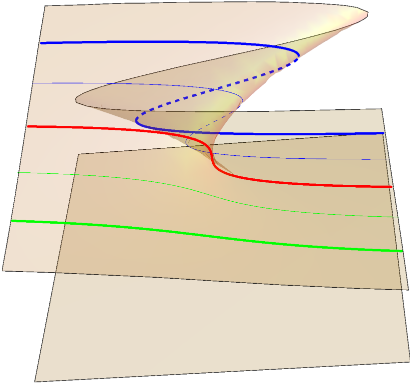
Zeeman's Machine - Study

Key Point

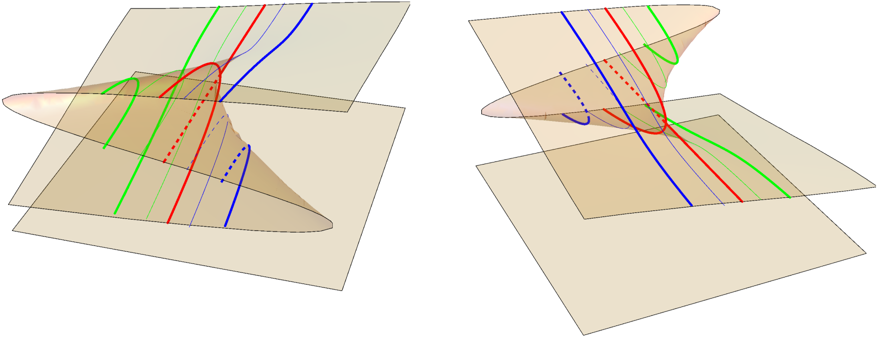
We want to know when the behaviour changes i.e. when a minimum appears/disappears.

Zeeman's Machine - Study





Zeeman's Machine - Study



Zeeman's Machine - Study

Definition

The **set of catastrophes**:

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Definition

Its projection over the plane xy defines the **bifurcation set**:

$$B_{\overline{V}} = \{(x, y) \in \mathbb{R}^2 \;/\; (x, y, \theta) \in C_{\overline{V}} \quad \text{for some } \theta\}$$

The projection $\chi_{\overline{V}} : C_{\overline{V}} \to B_{\overline{V}}$ is called **catastrophe germ.**

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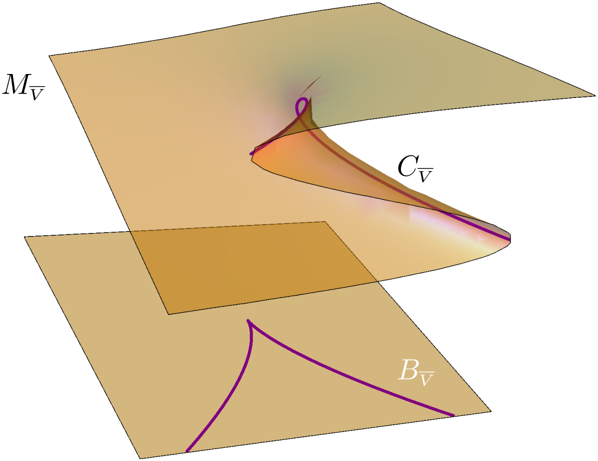
For the Zeeman's machine we obtain a cusp:

$$B_{\overline{V}} = \{(8\lambda^3, -6\lambda^2)\}$$

Zeeman's Machine - Study



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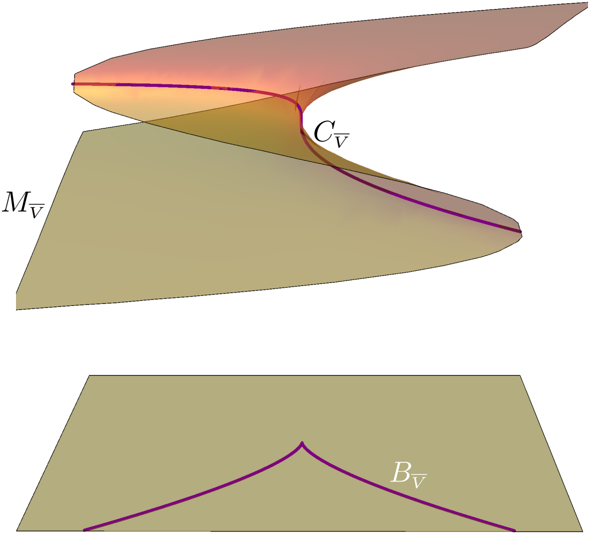

Zeeman's Machine - Study

Remarks

- The bifurcation set $B_{\overline{V}}$ is not a smooth manifold.

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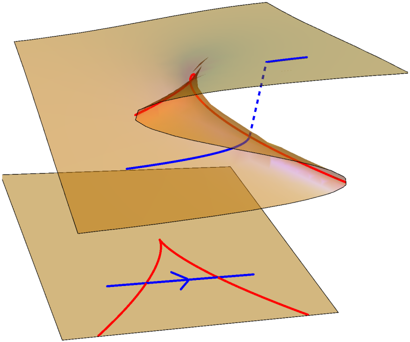
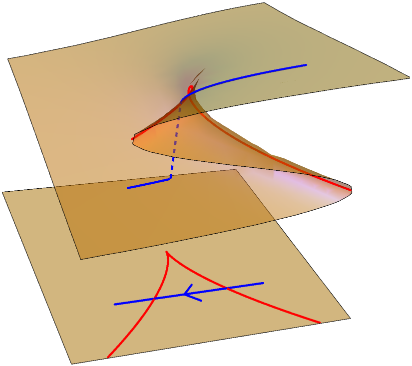
- The bifurcation set $B_{\overline{V}}$ is not a smooth manifold.
- $C_{\overline{V}}$ and $B_{\overline{V}}$ allow us to understand the pathologies of the Zeeman's catastrophe machine.

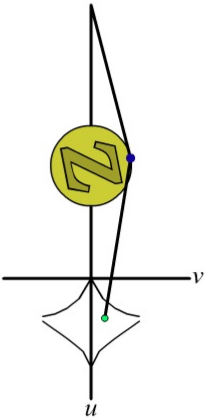
Zeeman's Machine - Study

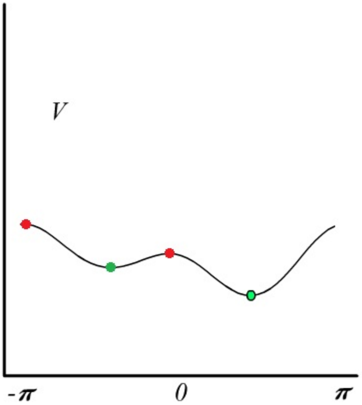




Zeeman's Machine - Study

Globally there are four linked cusps, with two minima and two maxima in the interior region.



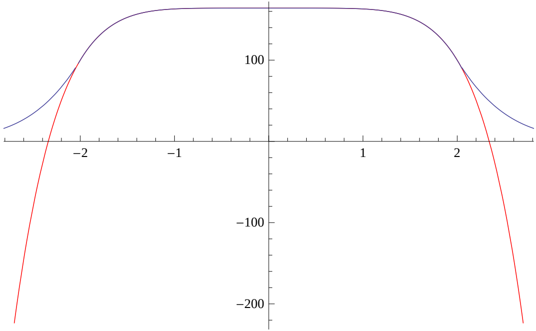


Part III: A bit of Theory

Theory - Germs

Definition

Two smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$ defined the same **germ** if they agree over some neighbourhood of the origin.



Germ - Equivalence of Germs

Definition

Dos germs f, g are equivalent if there exists $\varphi \in \mathcal{G}(n)$ such that $g = f \circ \varphi$, we denote it as $g \sim f$.

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Definition

Let $k \in \mathbb{N}$, we define the ***k*-jet** of a germ f as the k-truncated Taylor series at the origin:

$$j^k(f)(\bar{x}) = \sum_{\substack{\alpha=(\alpha_1 \ldots \alpha_n) \\ |\alpha| \leq k}} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(\bar{0})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \bar{x}^\alpha$$

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The elements of the subring:

$$\mathcal{M}(n) = \{g \in \mathcal{E}(n) \ / \ g(0) = 0\}$$

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have no inverse, and they form an ideal.

Remark

$\mathcal{M}(n)$ is the sole maximal ideal of the ring $\mathcal{E}(n)$, hence $\mathcal{E}(n)$ is a local ring.

Germ - Ideals

If we consider the product

$$\mathcal{M}(n)^k = \mathcal{M}(n) \cdot \overset{k)}{\cdots} \cdot \mathcal{M}(n)$$

it can be proven that:

$$\mathcal{M}(n)^k = \{g \in \mathcal{E}(n) \;/\; j^{k-1}(g) \equiv 0\}$$

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Remarks

- $\mathcal{M}(n)^{k+m} \subset \mathcal{M}(n)^k$
- If $f \in \mathcal{M}(n)^k$, then $\dfrac{\partial f}{\partial x_i} \in \mathcal{M}(n)^{k-1}$

Germ - Codimension of a Germ

Definition

We define the **Jacob's ideal of a germ** f as the ideal:

$$\Delta(f) = \left\{ g_1 \frac{\partial f}{\partial x_1} + \cdots + g_n \frac{\partial f}{\partial x_n} \;/\; g_i \in \mathcal{E}(n) \right\}$$

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Definition

We call **codimension of un germ** $f \in \mathcal{M}(n)^2$ a:

$$codim(f) = \dim \left(\mathcal{M}(n) \big/ \Delta(f) \right) \in \mathbb{N} \cup \{\infty\}$$

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Definition

The smallest K such that f is K-determined is its **determinative number** denoted $\sigma(f)$. If it does not exist, then we assign $\sigma(f) = \infty$.

Important Theorems

Theorem

Sea $f \in \mathcal{M}(n)^2 \implies \sigma(f) < \infty$ if and only if $codim(f) < \infty$

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Theorem

Let $f \in \mathcal{M}(n)^2$ such that $\sigma(f) < \infty$, then:

$$\sigma(f) \leq 2 + codim(f)$$

Theory - Unfolding

Definition

Let $f \in \mathcal{M}(n)^2$ be a germ, another germ $F \in \mathcal{M}(n+r)$ is a **r-unfolding of f** if $f(\bar{x}) = F(\bar{x}, \bar{0})$.

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Definition

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Remark

Two universal unfoldings of $f \in \mathcal{M}(n)^2$ (finite determined) are isomorphic.

Part IV: Algorithm for the construction of the Canonical Form

Algorithm

1. Let $F : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ be a smooth function $F(\bar{z}, \bar{p})$.

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2. Let $(\bar{z}_0, \bar{p}_0)$ such that $\bar{z}_0$ is a critical point of $F(\cdot, \bar{p}_0)$:

$$\frac{\partial F(\bar{z}_0, \bar{p}_0)}{\partial z_1} = 0 \qquad \cdots \qquad \frac{\partial F(\bar{z}_0, \bar{p}_0)}{\partial z_n} = 0$$

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3. We move to the origin:

$$\mathcal{F}(\bar{z}, \bar{p}) \equiv F(\bar{z} + \bar{z}_0, \bar{p} + \bar{p}_0) - F(\bar{z}_0, \bar{p}_0)$$

$\mathcal{F}$ satisfies $\mathcal{F}(\cdot, \bar{0})$ has $\bar{0}$ as a critical point and $\mathcal{F}(\bar{0}, \bar{0}) = 0$.

Algorithm

4. $\mathcal{F}(\bar{z}, \bar{p})$ is a unfolding of $f(\bar{z}) \equiv \mathcal{F}(\bar{z}, \bar{0})$. We assume that it is a universal unfolding (and $r = codim(f)$).

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5. $f(\bar{0}) = 0$ and $\bar{0}$ is a critical point of f, then $f \in \mathcal{M}(n)^2$.

6. If $\bar{0}$ is a **non degenerate** critical point, them by Morse lemma there exists $\varphi \in \mathcal{G}(n)$ such that:

$$f\Big(\varphi(\bar{z})\Big) = -z_1^2 - \cdots - z_k^2 + z_{k+1}^2 + \cdots + z_n^2$$

$k = ind(f)$. In a neighbourhood of the origin there are no more critical points of f. Besides $codim(f) = 0$.

Algorithm

7. Let us now assume that $\bar{0}$ is a **degenerated** critical point of f and $codim(f) \leq 5$, then there exists $\varphi \in \mathcal{G}(n)$ such that:

$$f\left(\varphi(\bar{z})\right) = \underbrace{-z_1^2 - \cdots - z_k^2 + z_{k+1}^2 + \cdots + z_\rho^2}_{q(\bar{z})} + Q$$

where $k = ind(f) \leq rg(f) \in \{n-2, n-1\}$ and Q is a polynomial of $(n - rg(f)) \in \{1, 2\}$ variables.

Q is one and only one of the 11 possible polynomials (we will see them later) and satisfies $codim(Q) = codim(f)$.

Algorithm

8. Once we have the unique Q we build a canonical universal unfolding $\overline{Q}$ of Q, with $r = codim(f)$ parameters.

   Hence $\overline{F} \equiv q + \overline{Q}$ is a universal unfolding of $q + Q = f \circ \varphi$.

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9. On the other hand $\mathcal{F}$ is a universal unfolding of f, then $\mathcal{F} \circ (\varphi \times Id_{\mathbb{R}^r})$ is a universal unfolding of $f \circ \varphi = q + Q$ with r parameters.

Algorithm

10. So we have $\overline{F}$ and $\mathcal{F} \circ (\varphi \times Id_r)$ universal unfoldings of f with the same number of parameters, they are this isomorphic. Furthermore, their catastrophe germ $\chi_{\mathcal{F}} \sim \chi_{\overline{F}}$ are equivalent.

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Important

The isomorphism between the universal unfoldings relates the initial data F with a canonical polynomial form $\overline{F} = q + \overline{Q}$, it relates also the equilibria, catastrophes and bifurcation sets.

Aplicación a la Máquina de Zeeman

Applying this algorithm to $F(x, y; \theta) = V_{(x,y)}(\theta)$ we obtain that there exists a isomorphism of universal unfoldings:

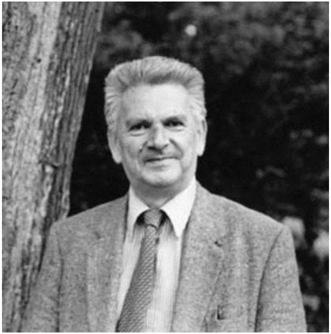
$$\begin{array}{l} \phi : \mathbb{R}^3 \to \mathbb{R}^3 \\ \overline{\phi} : \mathbb{R}^2 \to \mathbb{R}^2 \end{array} \qquad \varepsilon \in \mathcal{M}(2)$$

such that:

$$a_0\theta^4 + x\theta + y\theta^2 \equiv \overline{V}_{(x,y)}(\theta) =$$

$$= V_{(\bar{\phi}_1(x,y),\bar{\phi}_2(x,y)+y_1)}\Big(\varphi[\phi_1(\theta, x, y)]\Big) - V_{(0,y_1)}(0) + \varepsilon(x, y)$$

Thom Theorem



René Thom 1923 - 2002

Fields Medal
1958

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- the equilibria surface $M_g \subset \mathbb{R}^{n+r}$ is a r-dimensional smooth submanifold.
- $\chi_g : M_g \to \mathbb{R}^r$ is smooth and locally structurally stable for every equilibrium $(\bar{z}, \bar{p}) \in M_g$.

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Let $n \in \mathbb{N}$ and $1 \leq r \leq 5$, then there exists a dense open set $G \subset \mathcal{C}^{\infty}(\mathbb{R}^{n+r}, \mathbb{R})$ in the Whitney topology such that for every $g \in G$:

- the equilibria surface $M_g \subset \mathbb{R}^{n+r}$ is a r-dimensional smooth submanifold.
- $\chi_g : M_g \to \mathbb{R}^r$ is smooth and locally structurally stable for every equilibrium $(\bar{z}, \bar{p}) \in M_g$.

- Any initial potential V can be approximated by potentials $g \in G$ to study its catastrophes.
- Every good enough approximation are “equivalent”.

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- its catastrophe germ $\chi_g : M_g \to \mathbb{R}^r$ is equivalent to one of the 11 elementary catastrophes $\chi_h \times Id_{\mathbb{R}^{r-c}}$ for every $(\bar{z}, \bar{p}) \in C_g$.

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- Any initial potential V can be approximated by potentials $g \in G$ to study its catastrophes.
- Every good enough approximation are “equivalent”.
- For the Zeeman’s machine we saw that $\chi_V(C_V) = B_V$ was a cusp.

Thom Theorem - Classification

1) Fold $r = 1$

$$\overline{Q}_p(x) = x^3 + px$$

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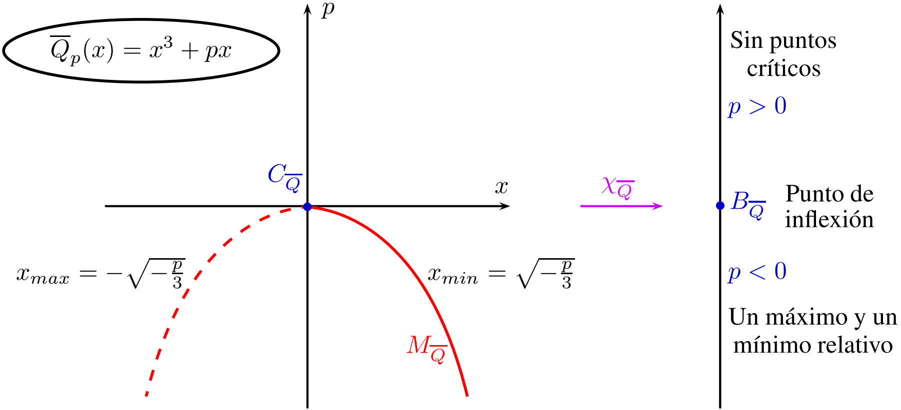
Projecting $C_{\overline{Q}}$ over the parameter space:

$$B_{\overline{Q}} = \{0\}$$

Thom Theorem - Classification



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2) Cusp $r = 2$

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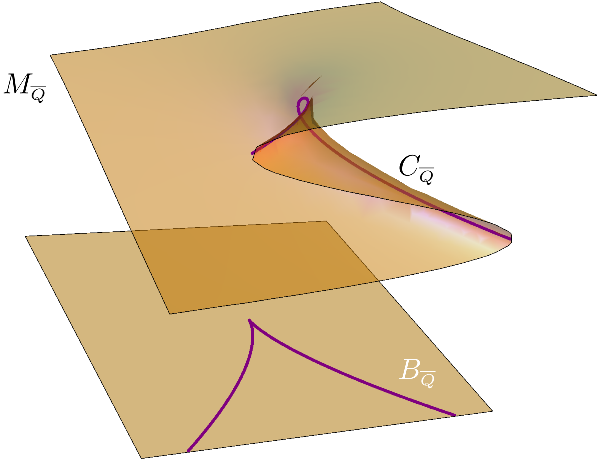
$\overline{Q}_{(p_1,p_2)}(x) = x^4 + p_1 x + p_2 x^2 (= \theta^4 + x\theta + y\theta^2$ Zeeman!)

$$M_{\overline{Q}} = \{(x, p_1, p_2) \in \mathbb{R}^3 \ / \ 4x^3 + p_1 + 2p_2 x = 0\}$$

$$C_{\overline{Q}} = \{(\lambda, 8\lambda^3, -6\lambda^2) \ / \ x \in \mathbb{R}\}$$

$$B_{\overline{Q}} = \{(8\lambda^3, -6\lambda^2) \ / \ x \in \mathbb{R}\}$$

Thom Theorem - Classification



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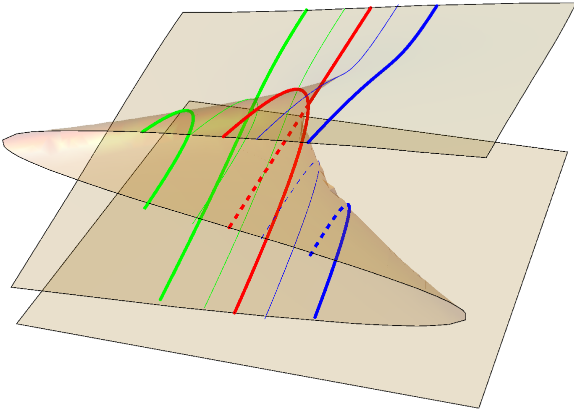

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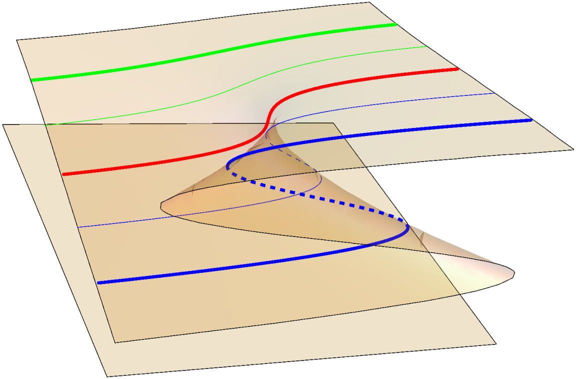
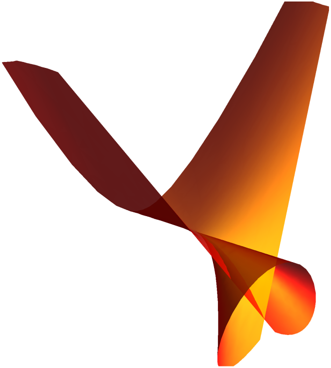


Thom Theorem - Classification

3) Swallowtail $r = 3$

$$\overline{Q}_{(p_1,p_2,p_3)}(x) = x^5 + p_1 x^3 + p_2 x^2 + p_3 x$$



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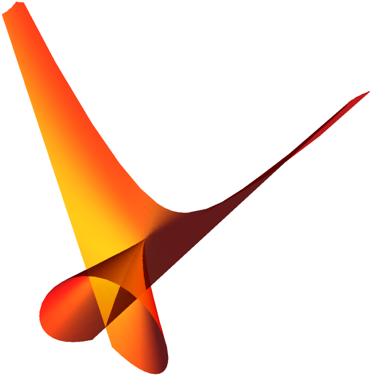


Thom Theorem - Classification

4) **Butterfly** $r = 4$

$$\overline{Q}_{(p_1,p_2,p_3,p_4)}(x) = x^6 + p_1 x^4 + p_2 x^3 + p_3 x^2 + p_4 x$$

5) **Indian Tent** $r = 5$

$$\overline{Q}_{(p_1,p_2,p_3,p_4,p_5)}(x) = x^7 + p_1 x^5 + p_2 x^4 + p_3 x^3 + p_4 x^2 + p_5 x$$

6) **Elliptic Umbilic** $r = 3$

$$\overline{Q}_{(p_1,p_2,p_3)}(x, y) = x^3 - xy^2 + p_1 y + p_2 x + p_3 y^2$$

7) **Hyperbolic Umbilic** $r = 3$

$$\overline{Q}_{(p_1,p_2,p_3)}(x, y) = x^3 + xy^2 + p_1 y + p_2 x + p_3 y^2$$

Thom Theorem - Classification

8) Parabolic Umbilic $r = 4$

$$\overline{Q}_{(p_1,p_2,p_3,p_4)}(x,y) = x^2y + y^4 + p_1x + p_2y + p_3x^2 + p_4y^2$$

9) Symbolic Umbilic $r = 5$

$$\overline{Q}_{(p_1,p_2,p_3,p_4,p_5)}(x,y) = x^3y + y^4 + p_1x + p_2y + p_3xy + p_4y^2 + p_5xy^2$$

10) Second Hyperbolic Umbilic $r = 5$

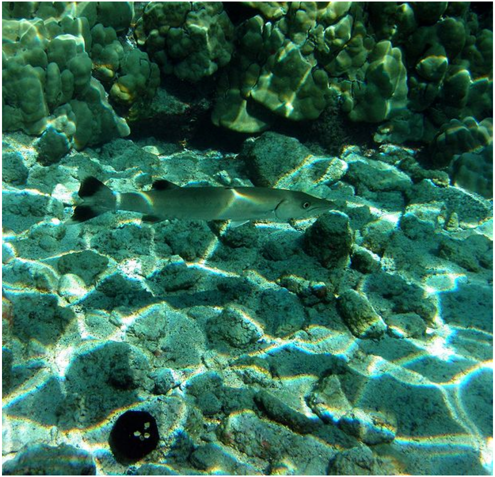
$$\overline{Q}_{(p_1,p_2,p_3,p_4,p_5)}(x,y) = x^2y + y^5 + p_1x + p_2y + p_3x^2 + p_4y^2 + p_5y^3$$

11) Second Elliptic Umbilic $r = 5$

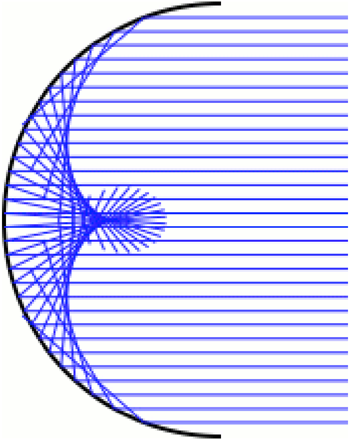
$$\overline{Q}_{(p_1,p_2,p_3,p_4,p_5)}(x,y) = x^2y - y^5 + p_1x + p_2y + p_3x^2 + p_4y^2 + p_5y^3$$

Part V: Cool Examples

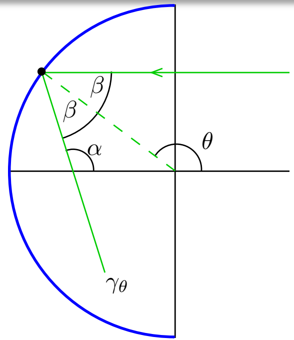
Caustics



Caustics



Caustics



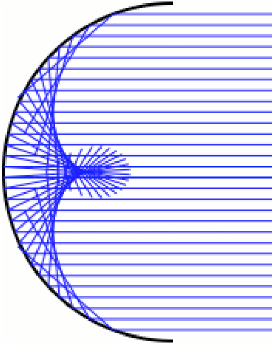
$$\alpha = 2\theta - \pi$$

$$\gamma_\theta : \; y - r\sin(\theta) = \tan(2\theta)(x - r\cos(\theta))$$

Differentiating and getting rid of θ:

$$\begin{cases} x = r\cos(\tau) - \dfrac{\cos^2(\tau)}{2}\Big(r\cos(\tau) + r\sin(\tau)\tan(2\tau)\Big) \\ y = r\sin(\tau) - \dfrac{\cos^2(\tau)}{2}\Big(r\cos(\tau) + r\sin(\tau)\tan(2\tau)\Big)\tan(2\tau) \end{cases}$$

Caustics



The shape of Planet Earth

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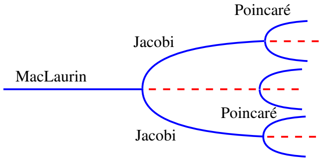

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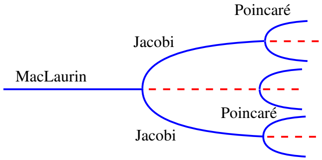
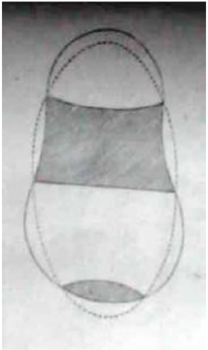


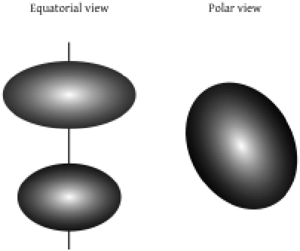
http://www.josleys.com/show_gallery.php?galid=313

Why *Poire* Shape?



Poincaré failed too!

The Dwarf Planet Haumea!



There exists a dwarf planet beyond the orbit of Neptune with Ellipsoidal shape.

- R. Thom, *Structural Stability and Morphogenesis*, Benjamin (1975).
- E.C. Zeeman, *Applications of the Catastrophe Theory*, Tokyo Int. Conf. on Manifolds (1973).

Thanks for your attention

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