Gauss words and topological classification of finitely determined map germs from $\mathbb{R}^2$ to $\mathbb{R}^2$

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joint work with

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1. Basic results and definitions

2. The link of a finitely determined map germ
   - Fukuda's theorem and the link of a map germ
   - Gauss words

3. Topological classification of corank 1 map germs

Definition (Set germ)

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. We say that two subsets $S, T \subseteq X$ *define the same germ at* x if there is an open neighborhood U of x in X such that $U \cap S = U \cap T$.

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Definition (Map germ)

Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be two topological spaces and $x \in X$. We will say that two maps $f : U \to Y$, $g : V \to Y$, where $U, V \subseteq X$ are open neighborhoods of x in X, *define the same germ at* x if there is another open neighborhood $W \subseteq U \cap V$ of x in X such that

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Once again, this relación define a binary equivalent relation and we call *map germ from* (X,x) *to* Y to each one of these classes. Moreover, if $f : U \to Y$ is a map, we will denote by $f : (X,x) \to Y$ the class defined by f and we will use the notation $f : (X,x) \to (Y,y)$ to denote a map germ $f : (X,x) \to Y$ such that $f(x) = y$.

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- From now on we suppose $X = Y = \mathbb{R}^2$ y $x = y = 0$.
- We will work with map germs of the form $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

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- We say that *f* is a germ of diffeomorphism if it has a representative that is a diffeomorphism.

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- We say that f and g are *$\mathcal{A}$-equivalent*, and we denote it by $f \sim_{\mathcal{A}} g$, if there are germs of diffeomorphism $\alpha : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\beta : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^2, 0) & \xrightarrow{\ f\ } & (\mathbb{R}^2, 0) \\ \downarrow \alpha & & \downarrow \beta \\ (\mathbb{R}^2, 0) & \xrightarrow{\ g\ } & (\mathbb{R}^2, 0) \end{array}$$

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- In the case that α and β are germs of homeomorphism, we will say that f and g are *topologically equivalent*.

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Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ be a smooth map germ. We will call k-jet of f and we will denote it by $j^k f(0)$ to the Taylor expansion of f of order k at $(0, 0)$.

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1. We will say that f is k-determined if for every smooth map germ $g : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ such that $j^k f(0) = j^k g(0)$ we have that f and g are $\mathcal{A}$-equivalent.

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2. f will be finitely determined if it is k-determined for some $k \geqslant 0$.

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Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be a smooth map. We say that f has a transversal double point if there are $p_1, p_2 \in \mathbb{R}$ such that $f(p_1) = f(p_2) = q$ y $(df)_{p_1}(T_{p_1}\mathbb{R}) + (df)_{p_2}(T_{p_2}\mathbb{R}) = T_q\mathbb{R}^2$

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Definition

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a smooth map germ such that $f^{-1}(0) = \{0\}$ and let $w \in \mathbb{R}^2$ a regular value small enough. We will call *topological degree of f* and we will denote it by $\deg(f)$ to

$$\deg(f) = \sum_{z_i \in f^{-1}(w)} \text{ind}(f, z_i),$$

where

$$\text{ind}(f, z_i) = \begin{cases} 1, \text{if } Jf(z_i) > 0, \\ -1, \text{if } Jf(z_i) < 0. \end{cases}$$

This definition doesn't depend on the chosen regular value w.

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We define the multiplicity of $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ as

$$m(f) = \dim_{\mathbb{R}} \frac{\mathbb{R}\{x, y\}}{\langle f_1, f_2 \rangle},$$

where f_1, f_2 denote the components of f and $\mathbb{R}\{x, y\}$ is the local algebra of germs of analytic functions $(\mathbb{R}^2, 0) \to \mathbb{R}$

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We will denote by $S_{1,0}(f)$ the set of fold points of f, by $S_{1,1}(f)$ the set of simple cusps and by $S_1(f) = S_{1,0}(f) \cup S_{1,1}(f)$ the set of singular points of corank 1 of f.

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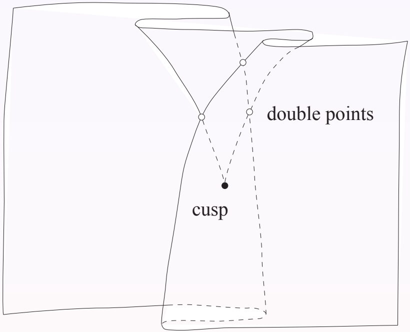
2. $f|_{S_{1,0}(f)}$ is an immersion with double transversal points.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Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ be a finitely determined map germ. The, there is a representative $f : U \to V$, with U, V being open subsets of $\mathbb{R}^2$ such that:

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1. $f^{-1}(0) = \{0\}$
2. $f : U \to V$ is proper
3. $f_{|U\setminus\{0\}}$ presents as unique singularities fold points and simple cusps and $f_{|(U\setminus\{0\})\cap S_{1,0}(f)}$ is an immersion with double transversal points.

Remark

If we choose an open neighborhood U in the representative of f small enough, we can leave out of it all the singular points of dimension 0 and as a consequence, $f_{|(U\setminus\{0\})\cap S_1(f)}$ will be an injective immersion.

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3. f is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.

Definition

We say that the stable map $f|_{\tilde{S}^1_\epsilon} : \tilde{S}^1_\epsilon \to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of Fukuda's theorem hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence. We also say that ϵ_0 is a Milnor-Fukuda radius for f.

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Corollary

Two finitely determined map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are topologically equivalent if their associated links are topologically equivalent.

Definition

Let $\gamma : S^1 \to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0 \in S^1$ in the source and $a_0 \in S^1$ in the target.

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Suppose that γ has r critical values labeled by r letters $a_1, \dots, a_r \in S^1$ and let us denote their inverse images by $z_1, \dots, z_k \in S^1$. We assume they are ordered such that $a_0 \leqslant a_1 < \cdots < a_r$ and $z_0 \leqslant z_1 < \cdots < z_k$ and following the orientation of each S^1.

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We define a map

$$\begin{array}{rccc} \sigma : & \{1,\dots,k\} & \longrightarrow & \{a_1,\dots,a_r,\overline{a}_1,\dots,\overline{a}_r\} \\ & i & \longmapsto & \begin{cases} a_j, & \text{if } \gamma(z_i) = a_j \text{ and } z_i \text{ is a regular point,} \\ \overline{a}_j, & \text{if } \gamma(z_i) = a_j \text{ and } z_i \text{ is a singular point.} \end{cases} \end{array}$$

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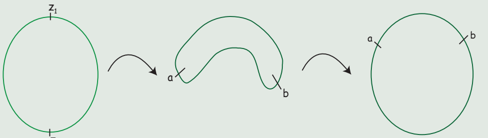
Suppose that γ has r critical values labeled by r letters $a_1, \dots, a_r \in S^1$ and let us denote their inverse images by $z_1, \dots, z_k \in S^1$. We assume they are ordered such that $a_0 \leqslant a_1 < \cdots < a_r$ and $z_0 \leqslant z_1 < \cdots < z_k$ and following the orientation of each S^1.
We define a map

$$\begin{array}{rccc} \sigma : & \{1, \dots, k\} & \longrightarrow & \{a_1, \dots, a_r, \overline{a}_1, \dots, \overline{a}_r\} \\ & i & \longmapsto & \begin{cases} a_j, & \text{if } \gamma(z_i) = a_j \text{ and } z_i \text{ is a regular point,} \\ \overline{a}_j, & \text{if } \gamma(z_i) = a_j \text{ and } z_i \text{ is a singular point.} \end{cases} \end{array}$$

We call *Gauss word* to the sequence $\sigma(1) \dots \sigma(k)$.

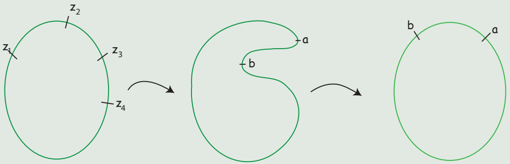
Examples

Fold



z_1z_2 $\overline{ab}$

Cusp



$z_1z_2z_3z_4$ $b\overline{ab}a$

(1) (2) (3)

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3. We have the following restrictions in its construction:
   - The number of folds (and as a consequence of different letters) have to be even.
   - Two inverse images of the same letter cannot appear together.

Theorem

Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

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Proposition

Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs such that they are topologically equivalent. Then, their respective links are topologically equivalent.

Corollary

Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then f, g are topologically equivalent if and only if

$$\begin{cases} w(f) \simeq w(g), & \text{if } f, g \text{ are singular outside the origin,} \\ |\deg(f)| = |\deg(g)|, & \text{if } f, g \text{ are regular outside the origin.} \end{cases}$$

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Remark

If f is regular outside the origin and $|deg(f)| = r$, then f is topologically equivalent to the germ $z \to z^r$, with $z = x + iy$.

Proposition

Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ be a map germ of corank $\leqslant 1$. Then f can be written as

$$f(x, y) = (x, f_2(x, y)),$$

where $f_2 : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$.

Let us consider now $j^1 f(0)$.

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If $j^1 f(0) \sim_{\mathcal{A}} (x, y)$, f is regular and we have finished. We suppose $j^1 f(0) \sim_{\mathcal{A}} (x, 0)$.

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Theorem

Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ be a finitely determined corank 1 map germ with $j^2 f(0) \sim_{\mathcal{A}} (x, y^2)$. Then f is topologically equivalent to (x, y^2).

Theorem

Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ be a finitely determined map germ such that $j^2 f(0) \sim_{\mathcal{A}} (x, xy)$ and let us denote by n the multiplicity of f, Then:

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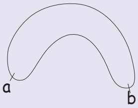
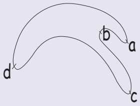
2. f is topologically equivalent to $(x, xy + y^3)$ (the simple cusp) if n is odd.

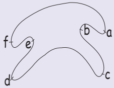
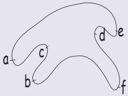
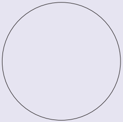
Theorem

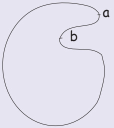
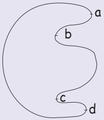
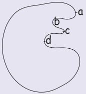
Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous finitely determined map germ, with 2-jet of type $(x, 0)$ and multiplicity $\leqslant 5$. Then, f is topologically equivalent to one of the germs of the following tables, depending on the topological configuration of its associated link and its topological degree.

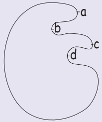
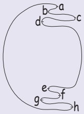
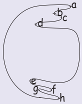
Theorem

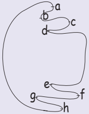
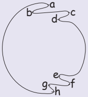
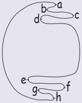
Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous finitely determined map germ, with 2-jet of type $(x, 0)$ and multiplicity $\leqslant 5$. Then, f is topologically equivalent to one of the germs of the following tables, depending on the topological configuration of its associated link and its topological degree.

Degree	Germ	Associated link
0	$(x, y^4 + x^2y^2)$	 $\overline{ab}$
	$(x, y^4 - xy^2 - x^2y)$	 $\overline{abcd}$

Degree	Germ	Associated link
0	$(x, y^4 - x^4y^2 + \frac{1}{4}x^6y)$	 $b\overline{ab}cbe\overline{de}fe$
	$(x, y^4 - x^2y^2 - \frac{1}{4}x^3y)$	 $c\overline{acb}cd\overline{fde}d$
1	$(x, y^3 + x^2y)$	 $\emptyset$

Degree	Germ	Associated link
1	$(x, xy + y^3)$	 $b\overline{ab}a$
	$(x, y^3 - x^2y)$	 $b\overline{ab}ad\overline{cd}c$
	$(x, y^5 + 2xy^3 + \frac{1}{2}x^2y)$	 $\overline{abcd}cadb$

Degree	Germ	Associated link
1	$(x, y^5 + 3xy^3 + 2x^2y)$	 $\overline{abcd}acbd$
	$(x, y^5 - 3x^2y^3 + \frac{5}{4}x^3y^2 + x^4y)$	 $db\overline{abcd}achf\overline{efgh}eg$
	$(x, y^5 - 5x^2y^3 + x^4y)$	 $db\overline{abcd}cahf\overline{efgh}ge$

Degree	Germ	Associated link
1	$(x, y^5 - \frac{3}{2}x^2y^3 + \frac{1}{2}x^4y)$	 $bd\overline{abcd}acfh\overline{efgh}eg$
	$(x, y^5 - 3x^2y^3 + 3x^4y)$	 $b\overline{ab}ad\overline{cd}cf\overline{ef}eh\overline{gh}g$
	$(x, y^5 - \frac{7}{2}x^4y^3 + 2x^6y^2 + x^8y)$	 $db\overline{abcd}acfh\overline{efgh}ge$

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¡Gracias por vuestra atención!

Gràcies per la vostra atenció!

Thanks for your attention!